Introduction to Quantum Mechanics HT 2010
Problems 4 (Weeks 1–2)

4.1 A particle is confined by the potential well

$$V(x) = \begin{cases} 0 & \text{for } |x| < a \\ \infty & \text{otherwise.} \end{cases} \tag{4.1}$$

Explain (a) why we can assume that there is a complete set of stationary states with well-defined parity and (b) why to find the stationary states we solve the TISE subject to the boundary condition $\psi(\pm a) = 0$.

Determine the particle's energy spectrum and give the wavefunctions of the first two stationary states.

4.2 At $t = 0$ the particle of Problem 4.1 has the wavefunction

$$\psi(x) = \begin{cases} 1/\sqrt{2a} & \text{for } |x| < a \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Find the probabilities that a measurement of its energy will yield: (a) $9\hbar^2\pi^2/(8ma^2)$; (b) $16\hbar^2\pi^2/(8ma^2)$.

4.3 Find the probability distribution of measuring momentum p for the particle described in Problem 4.2. Sketch and comment on your distribution. Hint: express $\langle p|x\rangle$ in the position representation.

4.4 Particles move in the potential

$$V(x) = \begin{cases} 0 & \text{for } x < 0 \\ V_0 & \text{for } x > 0. \end{cases} \tag{4.3}$$

Particles of mass m and energy $E > V_0$ are incident from $x = -\infty$. Show that the probability that a particle is reflected is

$$\left(\frac{k-K}{k+K}\right)^2, \tag{4.4}$$

where $k \equiv \sqrt{2mE}/\hbar$ and $K \equiv \sqrt{2m(E-V_0)}/\hbar$. Show directly from the TISE that the probability of transmission is

$$\frac{4kK}{(k+K)^2} \tag{4.5}$$

and check that the flux of particles moving away from the origin is equal to the incident particle flux.

4.5 Show that the energies of bound, odd-parity stationary states of the square potential well

$$V(x) = \begin{cases} 0 & \text{for } |x| < a \\ V_0 > 0 & \text{otherwise,} \end{cases} \tag{4.6}$$

are governed by

$$\cot(ka) = -\sqrt{\frac{W^2}{(ka)^2} - 1} \quad \text{where} \quad W \equiv \sqrt{\frac{2mV_0a^2}{\hbar^2}} \quad \text{and} \quad k^2 = 2mE/\hbar^2. \tag{4.7}$$

Show that for a bound odd-parity state to exist, we require $W > \pi/2$.

4.6 Give an example of a potential in which there is a complete set of bound stationary states of well-defined parity, and an alternative complete set of bound stationary states that are not eigenkets of the parity operator. Hint: modify the potential discussed apropos NH_3.

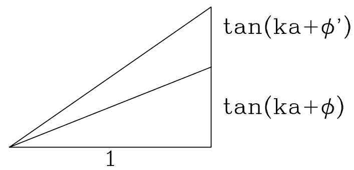

Figure 4.0 A triangle for Problem 5.9

4.7 A free particle of energy E approaches a square, one-dimensional potential well of depth V_0 and width $2a$. Show that the probability of being reflected by the well vanishes when $Ka = n\pi/2$, where n is an integer and $K = (2m(E+V_0)/\hbar^2)^{1/2}$. Explain this phenomenon in physical terms.

4.8 Show that the phase shifts ϕ (for the even-parity stationary state) and ϕ' (for the odd-parity state) that are associated with scattering by a classically allowed region of potential V_0 and width $2a$, satisfy

$$\tan(ka+\phi) = -(k/K)\cot(Ka) \quad \text{and} \quad \tan(ka+\phi') = (k/K)\tan(Ka),$$

where k and K are, respectively, the wavenumbers at infinity and in the scattering potential. Show that

$$P_{\rm refl} = \cos^2(\phi'-\phi) = \frac{(K/k - k/K)^2\sin^2(2Ka)}{(K/k+k/K)^2\sin^2(2Ka) + 4\cos^2(2Ka)}. \tag{4.8}$$

Hint: apply the cosine rule for an angle in a triangle in terms of the lengths of the triangle's sides to the top triangle in Figure 4.0.

4.9 A particle of energy E approaches from $x < 0$ a barrier in which the potential energy is $V(x) = V_\delta\delta(x)$. Show that the probability of its passing the barrier is

$$P_{\rm tun} = \frac{1}{1+(K/2k)^2} \quad \text{where} \quad k = \sqrt{\frac{2mE}{\hbar^2}}, \quad K = \frac{2mV_\delta}{\hbar^2}. \tag{4.9}$$

4.10 A system AB consists of two non-interacting parts A and B. The dynamical state of A is described by $|a\rangle$, and that of B by $|b\rangle$, so $|a\rangle$ satisfies the TDSE for A and similarly for $|b\rangle$. What is the ket describing the dynamical state of AB? In terms of the Hamiltonians $H_{\rm A}$ and $H_{\rm B}$ of the subsystems, write down the TDSE for the evolution of this ket and show that it is automatically satisfied. Do $H_{\rm A}$ and $H_{\rm B}$ commute? How is the TDSE changed when the subsystems are coupled by a small dynamical interaction $H_{\rm int}$? If A and B are harmonic oscillators, write down $H_{\rm A}$, $H_{\rm B}$. The oscillating particles are connected by a weak spring. Write down the appropriate form of the interaction Hamiltonian $H_{\rm int}$. Does $H_{\rm A}$ commute with $H_{\rm int}$? Explain the physical significance of your answer.

4.11 Explain what is implied by the statement that "the physical state of system A is correlated with the state of system B." Illustrate your answer by considering the momenta of cars on (i) the M25 at rush-hour, and (ii) the road over the Nullarbor Plain in southern Australia in the dead of night.

Explain why the states of A and B must be uncorrelated if it is possible to write the state of AB as a ket $|{\rm AB};\psi\rangle = |{\rm A};\psi_1\rangle|{\rm B};\psi_2\rangle$ that is a product of states of A and B. Given a complete set of states for A, $\{|{\rm A};i\rangle\}$ and a corresponding complete set of states for B, $\{|{\rm B};i\rangle\}$, write down an expression for a state of AB in which B is possibly correlated with A.