Creating Fields: Biot-Savart Law
Challenge Problem Solutions

Problem 1:
Find the magnetic field at point $P$ due to the following current distributions:

(a) The fields due to the straight wire segments are zero at $P$ because $d\mathbf{s}$ and $\mathbf{\hat{r}}$ are parallel or anti-parallel. For the field due to the arc segment, the magnitude of the magnetic field due to a differential current carrying element is given in this case by

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi} \frac{d\mathbf{s} \times \mathbf{\hat{r}}}{R^2} = \frac{\mu_0 I}{4\pi} \frac{IRd\theta(\sin \theta \mathbf{\hat{i}} - \cos \theta \mathbf{\hat{j}}) \times (-\cos \theta \mathbf{\hat{i}} - \sin \theta \mathbf{\hat{j}})}{R^2}.$$

Therefore,

$$\mathbf{B} = -\int_0^{\pi/2} \frac{\mu_0 I}{4\pi R} d\theta \mathbf{\hat{k}} = -\frac{\mu_0 I}{4\pi R} \left( \frac{\pi}{2} \right) \mathbf{\hat{k}} = -\left( \frac{\mu_0 I}{8R} \right) \mathbf{\hat{k}}$$

(b) There is no magnetic field due to the straight segments because point $P$ is along the lines. Using the general expression for $d\mathbf{B}$ obtained in (a), for the outer segment, we have

$$\mathbf{B}_{\text{out}} = \int_0^{\pi/2} \frac{\mu_0 I}{4\pi} \frac{d\theta}{b} \mathbf{\hat{k}} = -\left( \frac{\mu_0 I}{4b} \right) \mathbf{\hat{k}}.$$

Similarly, the contribution to the magnetic field from the inner segment is

$$\mathbf{B}_{\text{in}} = \int_0^\pi \frac{\mu_0 I}{4\pi} \frac{d\theta}{a} \mathbf{\hat{k}} = -\left( \frac{\mu_0 I}{4a} \right) \mathbf{\hat{k}}.$$

Therefore the net magnetic field at Point $P$ is
\[ \vec{B}_{\text{net}} = \vec{B}_{\text{out}} + \vec{B}_{\text{in}} = -\frac{\mu_0 I}{4} \left( \frac{1}{a} - \frac{1}{b} \right) \hat{k} \] (into the page since \( a < b \)).
Problem 2:

A conductor in the shape of a square loop of edge length $\ell = 0.400$ m carries a current $I = 10.0$ A as in the figure.

(a) Calculate the magnitude and direction of the magnetic field at the center of the square.

(b) If this conductor is formed into a single circular turn and carries the same current, what is the value of the magnetic field at the center?

Problem 2 Solutions:

For a finite wire carrying a current $I$, the contribution to the magnetic field at a point $P$ is given by Eq. (9.1.5) of the Course Notes:

$$B = \frac{\mu_0 I}{4\pi r} (\cos \theta_1 + \cos \theta_2)$$

where $\theta_1$ and $\theta_2$ are the angles which parameterize the length of the wire.

Consider the bottom segment. The cosine of the angles are given by

$$\cos \theta_2 = \cos \theta_1 = \cos 45^\circ = \frac{1}{\sqrt{2}}$$

This leads to
\[
B_1 = \frac{\mu_0 I}{4\pi(l/2)} \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) = \frac{\mu_0 I}{\sqrt{2}\pi l}
\]

The direction of \(\vec{B}_1\) is into the page. One may show that the other three segments yield the same contribution. Therefore, the total magnetic field at \(P\) is

\[
B = 4B_1 = 2\sqrt{2} \frac{\mu_0 I}{\pi l} = 2\sqrt{2} \frac{(4\pi \times 10^{-7} \text{T} \cdot \text{m}/\text{A})(10 \text{ A})}{\pi(0.40 \text{ m})} = 2.83 \times 10^{-5} \text{ T} \text{ (into the page)}
\]
Problem 3:

A wire is bent into the shape shown on the right, and the magnetic field is measured at \( P_1 \) when the current in the wire is \( I \).

From the discussion given in Example 9.1

The magnetic field is calculated as

\[
B = \frac{\mu_0 I}{4\pi l} (\cos \theta_2 + \cos \theta_1)
\]

For \( a \to b \), \( \theta_1 = \frac{\pi}{2} \) and \( \theta_2 = \frac{\pi}{4} \)

\[
B_{ab} = \frac{\mu_0 I}{4\pi l} \left( \frac{1}{\sqrt{2}} + 0 \right) = \frac{\sqrt{2} \mu_0 I}{8\pi R}
\]

For \( b \to c \), \( \theta_1 = \frac{\pi}{4} \) and \( \theta_2 = \frac{\pi}{4} \)

\[
B_{bc} = \frac{\mu_0 I}{4\pi l} \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) = \frac{\sqrt{2} \mu_0 I}{4\pi R}
\]

For \( c \to d \), \( \theta_1 = \frac{\pi}{4} \) and \( \theta_2 = \frac{\pi}{2} \)

\[
B_{cd} = \frac{\mu_0 I}{4\pi l} \left( 0 + \frac{1}{\sqrt{2}} \right) = \frac{\sqrt{2} \mu_0 I}{8\pi R}
\]

Therefore,

\[
B_1 = B_{ab} + B_{bc} + B_{cd} = \frac{\sqrt{2} \mu_0 I}{2\pi l} \quad \text{into page}
\]

The same segment of wire is then bent into a semi-circular shape shown in the figure below, and the magnetic field is measured at point \( P_2 \) when the current is again \( I \). If the total length of wire is the same in each case, what is the ratio of \( B_1 \) / \( B_2 \)?
Problem 3 Solution:

\[ \pi R = 4l \text{ or } R = \frac{4l}{\pi} \]

According to the Biot-Savart law, the magnitude of the magnetic field due to a differential current carrying element is given by

\[ dB = \frac{\mu_0 I}{4\pi} \frac{|d\vec{s} \times \vec{r}|}{r^2} = \frac{\mu_0 I}{4\pi} \frac{Rd\theta}{R^2} = \frac{\mu_0 I}{4\pi R} d\theta \]

Therefore,

\[ B_2 = \int_0^\pi \frac{\mu_0 I}{4\pi R} d\theta = \frac{\mu_0 I}{4\pi R} (\pi) = \frac{\mu_0 I}{4R} \frac{\pi}{16l} \text{ (into page)} \]

Hence,

\[ \frac{B_1}{B_2} = \left( \frac{\sqrt{2} \mu_0 I}{2\pi l} \right) \left( \frac{\pi \mu_0 I}{16l} \right) = \frac{16}{\sqrt{2\pi^2}} \approx 1.15 \]
Problem 4:

A wire carrying a current $I$ is bent into the shape of an exponential spiral, $r = e^{\theta}$, from $\theta = 0$ to $\theta = 2\pi$ as shown in the figure below.

To complete a loop, the ends of the spiral are connected by a straight wire along the $x$ axis. Find the magnitude and direction of $\mathbf{B}$ at the origin.

**Hint:** Use the Biot–Savart law. The angle $\beta$ between a radial line and its tangent line at any point on the curve $r = f(\theta)$ is related to the function in the following way:

$$\tan \beta = \frac{r}{dr/d\theta}$$

Thus in this case $r = e^{\theta}$, $\tan \beta = 1$ and $\beta = \pi/4$. Therefore, the angle between $d\hat{s}$ and $\hat{r}$ is $\pi - \beta = 3\pi/4$. Also

$$ds = \frac{dr}{\sin(\pi/4)} = \sqrt{2} \, dr$$

**Problem 4 Solution:**

There is no contribution from the straight portion of the wire since $d\hat{s} \times \hat{r} = 0$. For the field of the spiral, we apply Biot-Savart law:

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi} \frac{d\hat{s} \times \hat{r}}{r^2} = \frac{\mu_0 I}{4\pi} \frac{ds \sin \theta}{r^2} \hat{k}$$

$$= \frac{\mu_0 I}{4\pi} \frac{\sqrt{2} \, dr}{r^2} \hat{k} = \frac{\mu_0 I}{4\pi} \frac{dr}{r^2} \hat{k}$$

Substituting $r = e^{\theta}$ and $dr = e^{\theta} d\theta$, the above expression becomes

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi} \frac{e^{\theta} d\theta}{e^{2\theta}} \hat{k} = \frac{\mu_0 I}{4\pi} e^{-\theta} d\theta \hat{k}$$

Integrating the angle from $\theta$ to $2\pi$, we obtain
\[ \vec{B} = \frac{\mu_0 I}{4\pi} \hat{k} \int_0^{2\pi} e^{-i\theta} d\theta = \frac{\mu_0 I}{4\pi} (1 - e^{-2\pi}) \hat{k} \]
Problem 5:

A wire segment is bent into the shape of an Archimedes spiral (see sketch). The equation that describes the curve in the range $0 \leq \theta \leq \pi$ is

$$r(\theta) = a + \frac{b}{\pi} \theta, \text{ for } 0 \leq \theta \leq \pi$$

where $\theta$ is the angle from the $x$-axis in radians. The point $P$ is located at the origin of our $xy$ coordinate system. The vectors $\hat{e}_r$ and $\hat{e}_\theta$ are the unit vectors in the radial and azimuthal directions, respectively, as shown. The wire segment carries current $I$, flowing in the sense indicated.

What is the magnetic field at point $P$?

Problem 5 Solution:

We should begin by calculating the magnetic field due to a small current segment $ds$ (for example, at the location of the unit vectors on the above diagram). Using Biot Savart this creates a magnetic field:

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{s} \times \hat{r}}{r^2} = \frac{\mu_0 I}{4\pi} \frac{1}{r^2} \left( dr \hat{\theta} + r d\theta \hat{e}_\theta \right) \times (-\hat{e}_r) = \frac{\mu_0 I}{4\pi} \frac{r d\theta}{r^2} \hat{e}_\theta \times \hat{e}_\theta = \frac{\mu_0 I}{4\pi} \frac{r d\theta}{r^2} \hat{k}$$

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\theta}{r} \hat{k}$$

Now we just need to plug in $r(\theta)$ and integrate in order to find the field due to the entire spiral.

$$\vec{B} = \hat{k} \int_0^\pi \frac{\mu_0 I}{4\pi} \frac{d\theta}{r} = \hat{k} \int_0^\pi \frac{\mu_0 I}{4\pi} \left( \frac{d\theta}{a + \frac{b}{\pi} \theta} \right) = \hat{k} \frac{\mu_0 I}{4\pi} \frac{\pi}{b} \ln \left( a + \frac{b}{\pi} \right) \bigg|_a^b = \hat{k} \frac{\mu_0 I}{4\pi} \frac{1}{b} \left( \ln \left( 1 + \frac{b}{a} \right) \right)$$

where we made a simplification: $\ln \left( a + b \right) - \ln \left( a \right) = \ln \left( \frac{a+b}{a} \right) = \ln \left( 1 + \frac{b}{a} \right)$.
Of course, you should always do a reality check. In the limit that $b$ is small, we can use the approximation $\ln(1 + \frac{b}{a}) \cong \frac{b}{a}$ and our expression becomes $\bar{B} = \hat{k} \frac{\mu_o I}{4} \frac{1}{b \ a} = \hat{k} \frac{\mu_o I}{4a}$. This is what we expect because this is half the field at the center of a circle, and in the limit that $b$ goes to zero our spiral becomes a semi-circle.
Problem 6:

Four infinitely long parallel wires carrying equal current \( I \) are arranged in such a way that when looking at the cross section, they are at the corners of a square, as shown in the figure below. Currents in \( A \) and \( D \) point out of the page, and into the page at \( B \) and \( C \). What is the magnetic field at the center of the square?

Problem 6 Solution:

Four infinitely long parallel wires carrying equal current \( I \) are arranged in such a way that when looking at the cross section, they are at the corners of a square, as shown in the figure below. Currents in \( A \) and \( D \) point out of the page, and into the page at \( B \) and \( C \). What is the magnetic field at the center of the square?

The magnitude of the magnetic field a distance \( r \) from an infinite wire is

\[
B = \frac{\mu_0 I}{2\pi r}
\]

The direction of the field is azimuthal in a sense given by using the right hand rule. Thus, the magnetic field due to each wire at point \( P \) is

\[
\vec{B}_A = \hat{B}_A = \frac{\mu_0 I}{2\pi(a/\sqrt{2})} \left( -\frac{1}{\sqrt{2}} \hat{i} - \frac{1}{\sqrt{2}} \hat{j} \right)
\]

\[
\vec{B}_B = \hat{B}_B = \frac{\mu_0 I}{2\pi(a/\sqrt{2})} \left( \frac{1}{\sqrt{2}} \hat{i} - \frac{1}{\sqrt{2}} \hat{j} \right)
\]

\[
\vec{B}_C = \hat{B}_C = \frac{\mu_0 I}{2\pi(a/\sqrt{2})} \left( -\frac{1}{\sqrt{2}} \hat{i} - \frac{1}{\sqrt{2}} \hat{j} \right)
\]

\[
\vec{B}_D = \hat{B}_D = \frac{\mu_0 I}{2\pi(a/\sqrt{2})} \left( \frac{1}{\sqrt{2}} \hat{i} - \frac{1}{\sqrt{2}} \hat{j} \right)
\]

Adding up the individual contributions, we have

\[
\vec{B} = \vec{B}_A + \vec{B}_B + \vec{B}_C + \vec{B}_D = -\frac{2\mu_0 I}{\pi a} \hat{j}
\]