

Module 16, 17 and 18: The Magnetic Field¹

Table of Contents

8.1	Introduction.....	8-3
8.2	The Definition of a Magnetic Field	8-4
8.3	Magnetic Force on a Current-Carrying Wire.....	8-5
	Example 8.1: Magnetic Force on a Semi-Circular Loop	8-7
8.4	Torque on a Current Loop.....	8-9
	8.4.1 Magnetic force on a dipole	8-12
	8.4.2 Torques on a Dipole in a Constant Magnetic Field Animation.....	8-13
8.5	Charged Particles in a Uniform Magnetic Field	8-14
	8.5.1 Charged Particle Moving in a Uniform Magnetic Field Animation	8-16
8.6	Applications	8-16
	8.6.1 Velocity Selector	8-17
	8.6.2 Mass Spectrometer	8-18
8.7	Summary	8-19
8.8	Problem-Solving Tips	8-20
8.9	Solved Problems	8-21
	8.9.1 Rolling Rod	8-21
	8.9.2 Suspended Conducting Rod	8-22
	8.9.3 Charged Particles in Magnetic Field	8-23
	8.9.4 Bar Magnet in Non-Uniform Magnetic Field.....	8-24
8.10	Conceptual Questions	8-25
8.11	Additional Problems	8-25
	8.11.1 Force Exerted by a Magnetic Field	8-25
	8.11.2 Magnetic Force on a Current Carrying Wire.....	8-25
	8.11.3 Sliding Bar.....	8-26
	8.11.4 Particle Trajectory	8-27
	8.11.5 Particle Orbits in a Magnetic Field.....	8-27
	8.11.6 Force and Torque on a Current Loop	8-28
	8.11.7 Force on a Wire	8-28
	8.11.8 Levitating Wire.....	8-29

¹ These notes are excerpted “Introduction to Electricity and Magnetism” by Sen-Ben Liao, Peter Dourmashkin, and John Belcher, Copyright 2004, ISBN 0-536-81207-1.

Introduction to Magnetic Fields

8.1 Introduction

We have seen that a charged object produces an electric field \vec{E} at all points in space. In a similar manner, a bar magnet is a source of a magnetic field \vec{B} . This can be readily demonstrated by moving a compass near the magnet. The compass needle will line up along the direction of the magnetic field produced by the magnet, as depicted in Figure 8.1.1.

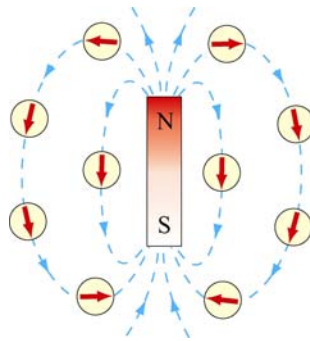


Figure 8.1.1 Magnetic field produced by a bar magnet

Notice that the bar magnet consists of two poles, which are designated as the north (N) and the south (S). Magnetic fields are strongest at the poles. The magnetic field lines leave from the north pole and enter the south pole. When holding two bar magnets close to each other, the like poles will repel each other while the opposite poles attract (Figure 8.1.2).

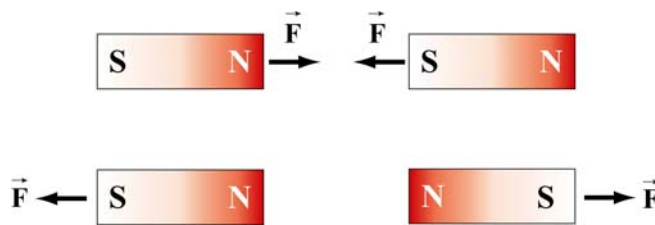


Figure 8.1.2 Magnets attracting and repelling

Unlike electric charges which can be isolated, the two magnetic poles always come in a pair. When you break the bar magnet, two new bar magnets are obtained, each with a north pole and a south pole (Figure 8.1.3). In other words, magnetic “monopoles” do not exist in isolation, although they are of theoretical interest.



Figure 8.1.3 Magnetic monopoles do not exist in isolation

How do we define the magnetic field $\vec{\mathbf{B}}$? In the case of an electric field $\vec{\mathbf{E}}$, we have already seen that the field is defined as the force per unit charge:

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}_e}{q} \quad (8.1.1)$$

However, due to the absence of magnetic monopoles, $\vec{\mathbf{B}}$ must be defined in a different way.

8.2 The Definition of a Magnetic Field

To define the magnetic field at a point, consider a particle of charge q and moving at a velocity $\vec{\mathbf{v}}$. Experimentally we have the following observations:

- (1) The magnitude of the magnetic force $\vec{\mathbf{F}}_B$ exerted on the charged particle is proportional to both v and q .
- (2) The magnitude and direction of $\vec{\mathbf{F}}_B$ depends on $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$.
- (3) The magnetic force $\vec{\mathbf{F}}_B$ vanishes when $\vec{\mathbf{v}}$ is parallel to $\vec{\mathbf{B}}$. However, when $\vec{\mathbf{v}}$ makes an angle θ with $\vec{\mathbf{B}}$, the direction of $\vec{\mathbf{F}}_B$ is perpendicular to the plane formed by $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$, and the magnitude of $\vec{\mathbf{F}}_B$ is proportional to $\sin \theta$.
- (4) When the sign of the charge of the particle is switched from positive to negative (or vice versa), the direction of the magnetic force also reverses.

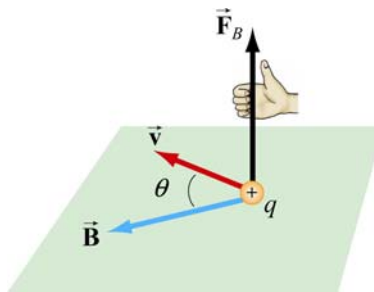


Figure 8.2.1 The direction of the magnetic force

The above observations can be summarized with the following equation:

$$\vec{\mathbf{F}}_B = q\vec{\mathbf{v}} \times \vec{\mathbf{B}} \quad (8.2.1)$$

The above expression can be taken as the working definition of the magnetic field at a point in space. The magnitude of \vec{F}_B is given by

$$F_B = |q| vB \sin \theta \quad (8.2.2)$$

The SI unit of magnetic field is the tesla (T):

$$1 \text{ tesla} = 1 \text{ T} = 1 \frac{\text{Newton}}{(\text{Coulomb})(\text{meter/second})} = 1 \frac{\text{N}}{\text{C} \cdot \text{m/s}} = 1 \frac{\text{N}}{\text{A} \cdot \text{m}}$$

Another commonly used non-SI unit for \vec{B} is the *gauss* (G), where $1\text{T} = 10^4 \text{ G}$.

Note that \vec{F}_B is always perpendicular to \vec{v} and \vec{B} , and cannot change the particle's speed v (and thus the kinetic energy). In other words, magnetic force cannot speed up or slow down a charged particle. Consequently, \vec{F}_B can do no work on the particle:

$$dW = \vec{F}_B \cdot d\vec{s} = q(\vec{v} \times \vec{B}) \cdot \vec{v} dt = q(\vec{v} \times \vec{v}) \cdot \vec{B} dt = 0 \quad (8.2.3)$$

The direction of \vec{v} , however, can be altered by the magnetic force, as we shall see below.

8.3 Magnetic Force on a Current-Carrying Wire

We have just seen that a charged particle moving through a magnetic field experiences a magnetic force \vec{F}_B . Since electric current consists of a collection of charged particles in motion, when placed in a magnetic field, a current-carrying wire will also experience a magnetic force.

Consider a long straight wire suspended in the region between the two magnetic poles. The magnetic field points out the page and is represented with dots (\bullet). It can be readily demonstrated that when a downward current passes through, the wire is deflected to the left. However, when the current is upward, the deflection is rightward, as shown in Figure 8.3.1.

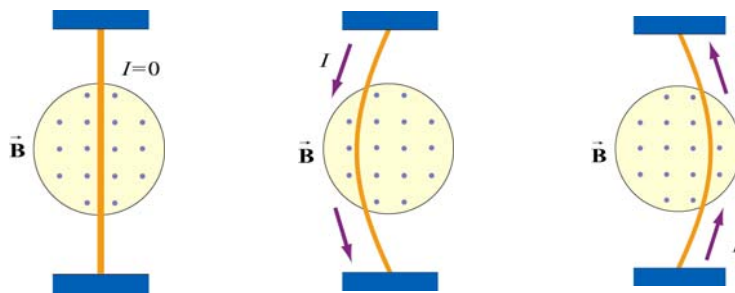


Figure 8.3.1 Deflection of current-carrying wire by magnetic force

To calculate the force exerted on the wire, consider a segment of wire of length ℓ and cross-sectional area A , as shown in Figure 8.3.2. The magnetic field points into the page, and is represented with crosses (X).

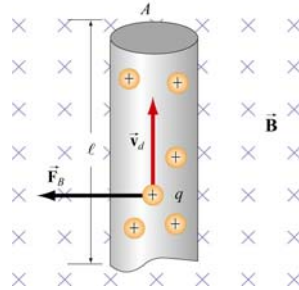


Figure 8.3.2 Magnetic force on a conducting wire

The charges move at an average drift velocity \vec{v}_d . Since the total amount of charge in this segment is $Q_{\text{tot}} = q(nA\ell)$, where n is the number of charges per unit volume, the total magnetic force on the segment is

$$\vec{F}_B = Q_{\text{tot}} \vec{v}_d \times \vec{B} = qnA\ell(\vec{v}_d \times \vec{B}) = I(\vec{\ell} \times \vec{B}) \quad (8.3.1)$$

where $I = nqv_d A$, and $\vec{\ell}$ is a *length vector* with a magnitude ℓ and directed along the direction of the electric current.

For a wire of arbitrary shape, the magnetic force can be obtained by summing over the forces acting on the small segments that make up the wire. Let the differential segment be denoted as $d\vec{s}$ (Figure 8.3.3).

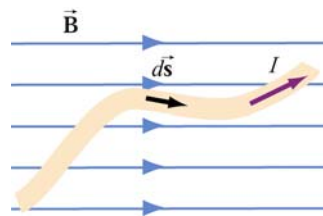


Figure 8.3.3 Current-carrying wire placed in a magnetic field

The magnetic force acting on the segment is

$$d\vec{F}_B = Id\vec{s} \times \vec{B} \quad (8.3.2)$$

Thus, the total force is

$$\boxed{\vec{F}_B = I \int_a^b d\vec{s} \times \vec{B}} \quad (8.3.3)$$

where a and b represent the endpoints of the wire.

As an example, consider a curved wire carrying a current I in a uniform magnetic field $\vec{\mathbf{B}}$, as shown in Figure 8.3.4.

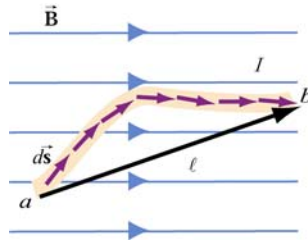


Figure 8.3.4 A curved wire carrying a current I .

Using Eq. (8.3.3), the magnetic force on the wire is given by

$$\vec{\mathbf{F}}_B = I \left(\int_a^b d\vec{\mathbf{s}} \right) \times \vec{\mathbf{B}} = I \vec{\mathbf{l}} \times \vec{\mathbf{B}} \quad (8.3.4)$$

where $\vec{\mathbf{l}}$ is the length vector directed from a to b . However, if the wire forms a closed loop of arbitrary shape (Figure 8.3.5), then the force on the loop becomes

$$\vec{\mathbf{F}}_B = I \left(\oint d\vec{\mathbf{s}} \right) \times \vec{\mathbf{B}} \quad (8.3.5)$$

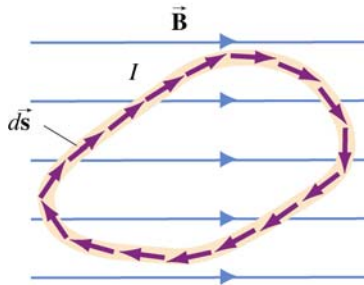


Figure 8.3.5 A closed loop carrying a current I in a uniform magnetic field.

Since the set of differential length elements $d\vec{\mathbf{s}}$ form a closed polygon, and their vector sum is zero, i.e., $\oint d\vec{\mathbf{s}} = 0$. The net magnetic force on a closed loop is $\vec{\mathbf{F}}_B = \vec{\mathbf{0}}$.

Example 8.1: Magnetic Force on a Semi-Circular Loop

Consider a closed semi-circular loop lying in the xy plane carrying a current I in the counterclockwise direction, as shown in Figure 8.3.6.

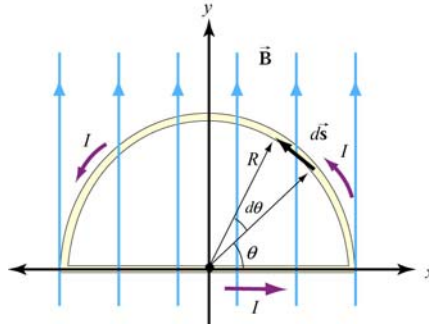


Figure 8.3.6 Semi-circular loop carrying a current I

A uniform magnetic field pointing in the $+y$ direction is applied. Find the magnetic force acting on the straight segment and the semicircular arc.

Solution:

Let $\vec{B} = B\hat{j}$ and \vec{F}_1 and \vec{F}_2 the forces acting on the straight segment and the semicircular parts, respectively. Using Eq. (8.3.3) and noting that the length of the straight segment is $2R$, the magnetic force is

$$\vec{F}_1 = I(2R\hat{i}) \times (B\hat{j}) = 2IRB\hat{k}$$

where \hat{k} is directed out of the page.

To evaluate \vec{F}_2 , we first note that the differential length element $d\vec{s}$ on the semicircle can be written as $d\vec{s} = ds\hat{\theta} = Rd\theta(-\sin\theta\hat{i} + \cos\theta\hat{j})$. The force acting on the length element $d\vec{s}$ is

$$d\vec{F}_2 = Id\vec{s} \times \vec{B} = IRd\theta(-\sin\theta\hat{i} + \cos\theta\hat{j}) \times (B\hat{j}) = -IBR\sin\theta d\theta\hat{k}$$

Here we see that $d\vec{F}_2$ points into the page. Integrating over the entire semi-circular arc, we have

$$\vec{F}_2 = -IBR\hat{k} \int_0^\pi \sin\theta d\theta = -2IBR\hat{k}$$

Thus, the net force acting on the semi-circular wire is

$$\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2 = \vec{0}$$

This is consistent from our previous claim that the net magnetic force acting on a closed current-carrying loop must be zero.

8.4 Torque on a Current Loop

What happens when we place a rectangular loop carrying a current I in the xy plane and switch on a uniform magnetic field $\vec{\mathbf{B}} = B\hat{\mathbf{i}}$ which runs parallel to the plane of the loop, as shown in Figure 8.4.1(a)?

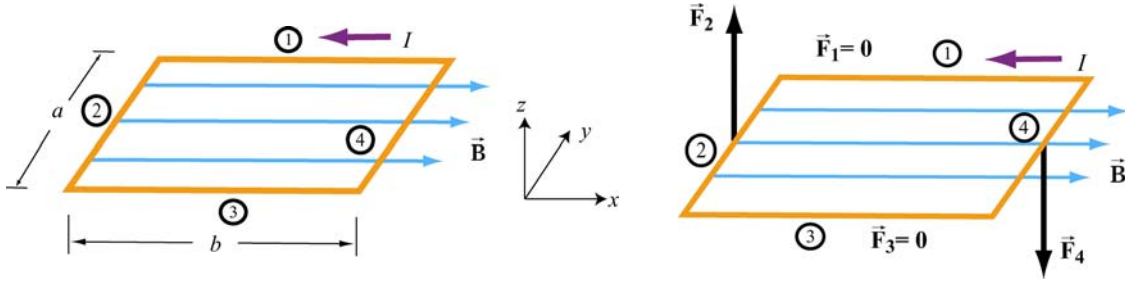


Figure 8.4.1 (a) A rectangular current loop placed in a uniform magnetic field. (b) The magnetic forces acting on sides 2 and 4.

From Eq. 8.4.1, we see the magnetic forces acting on sides 1 and 3 vanish because the length vectors $\vec{\ell}_1 = -b\hat{\mathbf{i}}$ and $\vec{\ell}_3 = b\hat{\mathbf{i}}$ are parallel and anti-parallel to $\vec{\mathbf{B}}$ and their cross products vanish. On the other hand, the magnetic forces acting on segments 2 and 4 are non-vanishing:

$$\begin{cases} \vec{\mathbf{F}}_2 = I(-a\hat{\mathbf{j}}) \times (B\hat{\mathbf{i}}) = IaB\hat{\mathbf{k}} \\ \vec{\mathbf{F}}_4 = I(a\hat{\mathbf{j}}) \times (B\hat{\mathbf{i}}) = -IaB\hat{\mathbf{k}} \end{cases} \quad (8.4.1)$$

with $\vec{\mathbf{F}}_2$ pointing out of the page and $\vec{\mathbf{F}}_4$ into the page. Thus, the net force on the rectangular loop is

$$\vec{\mathbf{F}}_{\text{net}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \vec{\mathbf{F}}_3 + \vec{\mathbf{F}}_4 = \vec{\mathbf{0}} \quad (8.4.2)$$

as expected. Even though the net force on the loop vanishes, the forces $\vec{\mathbf{F}}_2$ and $\vec{\mathbf{F}}_4$ will produce a torque which causes the loop to rotate about the y -axis (Figure 8.4.2). The torque with respect to the center of the loop is

$$\begin{aligned} \vec{\boldsymbol{\tau}} &= \left(-\frac{b}{2}\hat{\mathbf{i}}\right) \times \vec{\mathbf{F}}_2 + \left(\frac{b}{2}\hat{\mathbf{i}}\right) \times \vec{\mathbf{F}}_4 = \left(-\frac{b}{2}\hat{\mathbf{i}}\right) \times (IaB\hat{\mathbf{k}}) + \left(\frac{b}{2}\hat{\mathbf{i}}\right) \times (-IaB\hat{\mathbf{k}}) \\ &= \left(\frac{IabB}{2} + \frac{IabB}{2}\right)\hat{\mathbf{j}} = IabB\hat{\mathbf{j}} = IAB\hat{\mathbf{j}} \end{aligned} \quad (8.4.3)$$

where $A = ab$ represents the area of the loop and the positive sign indicates that the rotation is clockwise about the y -axis. It is convenient to introduce the area vector $\vec{A} = A\hat{n}$ where \hat{n} is a unit vector in the direction normal to the plane of the loop. The direction of the positive sense of \hat{n} is set by the conventional right-hand rule. In our case, we have $\hat{n} = +\hat{k}$. The above expression for torque can then be rewritten as

$$\vec{\tau} = I\vec{A} \times \vec{B} \quad (8.4.4)$$

Notice that the magnitude of the torque is at a maximum when \vec{B} is parallel to the plane of the loop (or perpendicular to \vec{A}).

Consider now the more general situation where the loop (or the area vector \vec{A}) makes an angle θ with respect to the magnetic field.

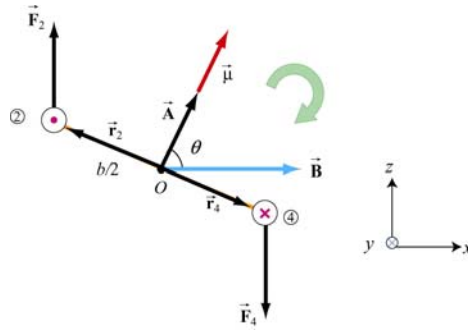


Figure 8.4.2 Rotation of a rectangular current loop

From Figure 8.4.2, the lever arms and can be expressed as:

$$\vec{r}_2 = \frac{b}{2}(-\sin\theta\hat{i} + \cos\theta\hat{k}) = -\vec{r}_4 \quad (8.4.5)$$

and the net torque becomes

$$\begin{aligned} \vec{\tau} &= \vec{r}_2 \times \vec{F}_2 + \vec{r}_4 \times \vec{F}_4 = 2\vec{r}_2 \times \vec{F}_2 = 2 \cdot \frac{b}{2}(-\sin\theta\hat{i} + \cos\theta\hat{k}) \times (IaB\hat{k}) \\ &= IabB \sin\theta \hat{j} = I\vec{A} \times \vec{B} \end{aligned} \quad (8.4.6)$$

For a loop consisting of N turns, the magnitude of the torque is

$$\tau = NIAB \sin\theta \quad (8.4.7)$$

The quantity $NI\vec{A}$ is called the magnetic dipole moment $\vec{\mu}$:

$$\boxed{\vec{\mu} = NI\vec{A}} \quad (8.4.8)$$

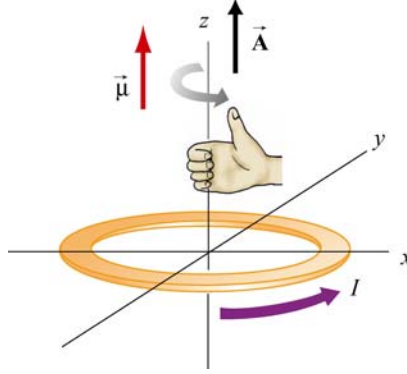


Figure 8.4.3 Right-hand rule for determining the direction of $\vec{\mu}$

The direction of $\vec{\mu}$ is the same as the area vector \vec{A} (perpendicular to the plane of the loop) and is determined by the right-hand rule (Figure 8.4.3). The SI unit for the magnetic dipole moment is ampere-meter² ($\text{A} \cdot \text{m}^2$). Using the expression for $\vec{\mu}$, the torque exerted on a current-carrying loop can be rewritten as

$$\boxed{\vec{\tau} = \vec{\mu} \times \vec{B}} \quad (8.4.9)$$

The above equation is analogous to $\vec{\tau} = \vec{p} \times \vec{E}$ in Eq. (2.8.3), the torque exerted on an electric dipole moment \vec{p} in the presence of an electric field \vec{E} . Recalling that the potential energy for an electric dipole is $U = -\vec{p} \cdot \vec{E}$ [see Eq. (2.8.7)], a similar form is expected for the magnetic case. The work done by an external agent to rotate the magnetic dipole from an angle θ_0 to θ is given by

$$\begin{aligned} W_{\text{ext}} &= \int_{\theta_0}^{\theta} \tau d\theta' = \int_{\theta_0}^{\theta} (\mu B \sin \theta') d\theta' = \mu B (\cos \theta_0 - \cos \theta) \\ &= \Delta U = U - U_0 \end{aligned} \quad (8.4.10)$$

Once again, $W_{\text{ext}} = -W$, where W is the work done by the magnetic field. Choosing $U_0 = 0$ at $\theta_0 = \pi/2$, the dipole in the presence of an external field then has a potential energy of

$$U = -\mu B \cos \theta = -\vec{\mu} \cdot \vec{B} \quad (8.4.11)$$

The configuration is at a stable equilibrium when $\vec{\mu}$ is aligned parallel to \vec{B} , making U a minimum with $U_{\text{min}} = -\mu B$. On the other hand, when $\vec{\mu}$ and \vec{B} are anti-parallel, $U_{\text{max}} = +\mu B$ is a maximum and the system is unstable.

8.4.1 Magnetic force on a dipole

As we have shown above, the force experienced by a current-carrying rectangular loop (i.e., a magnetic dipole) placed in a uniform magnetic field is zero. What happens if the magnetic field is non-uniform? In this case, there will be a net force acting on the dipole.

Consider the situation where a small dipole $\vec{\mu}$ is placed along the symmetric axis of a bar magnet, as shown in Figure 8.4.4.

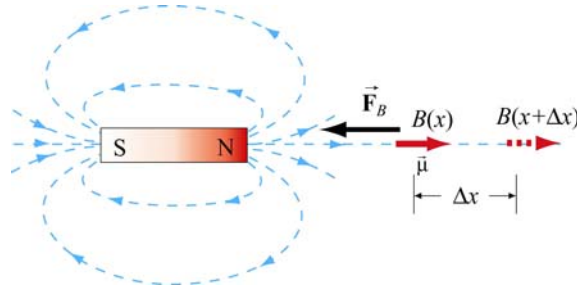


Figure 8.4.4 A magnetic dipole near a bar magnet.

The dipole experiences an attractive force by the bar magnet whose magnetic field is non-uniform in space. Thus, an external force must be applied to move the dipole to the right. The amount of force F_{ext} exerted by an external agent to move the dipole by a distance Δx is given by

$$F_{\text{ext}} \Delta x = W_{\text{ext}} = \Delta U = -\mu B(x + \Delta x) + \mu B(x) = -\mu [B(x + \Delta x) - B(x)] \quad (8.4.12)$$

where we have used Eq. (8.4.11). For small Δx , the external force may be obtained as

$$F_{\text{ext}} = -\mu \frac{[B(x + \Delta x) - B(x)]}{\Delta x} = -\mu \frac{dB}{dx} \quad (8.4.13)$$

which is a positive quantity since $dB/dx < 0$, i.e., the magnetic field decreases with increasing x . This is precisely the force needed to overcome the attractive force due to the bar magnet. Thus, we have

$$F_B = \mu \frac{dB}{dx} = \frac{d}{dx} (\vec{\mu} \cdot \vec{B}) \quad (8.4.14)$$

More generally, the magnetic force experienced by a dipole $\vec{\mu}$ placed in a non-uniform magnetic field \vec{B} can be written as

$$\vec{F}_B = \nabla (\vec{\mu} \cdot \vec{B}) \quad (8.4.15)$$

where

$$\nabla = \frac{\partial}{\partial x} \hat{\mathbf{i}} + \frac{\partial}{\partial y} \hat{\mathbf{j}} + \frac{\partial}{\partial z} \hat{\mathbf{k}} \quad (8.4.16)$$

is the gradient operator.

8.4.2 Torques on a Dipole in a Constant Magnetic Field Animation

“...To understand this point, we have to consider that a [compass] needle vibrates by gathering upon itself, because of its magnetic condition and polarity, a certain amount of the lines of force, which would otherwise traverse the space about it...”

Michael Faraday [1855]

Consider a magnetic dipole in a constant background field. Historically, we note that Faraday understood the oscillations of a compass needle in exactly the way we describe here. We show in Figure 8.4.5 a magnetic dipole in a “dip needle” oscillating in the magnetic field of the Earth, at a latitude approximately the same as that of Boston. The magnetic field of the Earth is predominantly downward and northward at these Northern latitudes, as the visualization indicates.

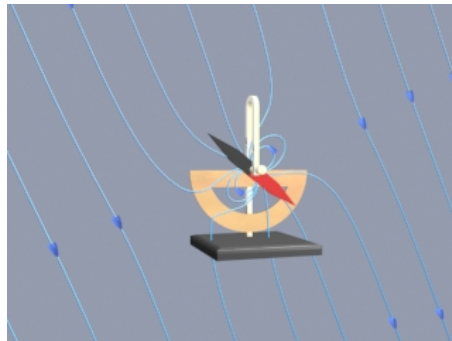


Figure 8.4.5 A magnetic dipole in the form of a dip needle oscillates in the magnetic field of the Earth ([link](#))

To explain what is going on in this visualization, suppose that the magnetic dipole vector is initially along the direction of the earth’s field and rotating clockwise. As the dipole rotates, the magnetic field lines are compressed and stretched. The tensions and pressures associated with this field line stretching and compression results in an electromagnetic torque on the dipole that slows its clockwise rotation. Eventually the dipole comes to rest. But the counterclockwise torque still exists, and the dipole then starts to rotate counterclockwise, passing back through being parallel to the Earth’s field again (where the torque goes to zero), and overshooting.

As the dipole continues to rotate counterclockwise, the magnetic field lines are now compressed and stretched in the opposite sense. The electromagnetic torque has reversed sign, now slowing the dipole in its counterclockwise rotation. Eventually the dipole will come to rest, start rotating clockwise once more, and pass back through being parallel to

the field, as in the beginning. If there is no damping in the system, this motion continues indefinitely.

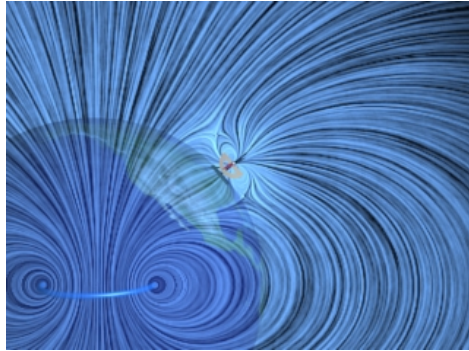


Figure 8.4.6 A magnetic dipole in the form of a dip needle rotates oscillates in the magnetic field of the Earth. We show the currents that produce the earth's field in this visualization ([link](#))

What about the conservation of angular momentum in this situation? Figure 8.4.6 shows a global picture of the field lines of the dip needle and the field lines of the Earth, which are generated deep in the core of the Earth. If you examine the stresses transmitted between the Earth and the dip needle in this visualization, you can convince yourself that any clockwise torque on the dip needle is accompanied by a counterclockwise torque on the currents producing the earth's magnetic field. Angular momentum is conserved by the exchange of equal and opposite amounts of angular momentum between the compass and the currents in the Earth's core.

8.5 Charged Particles in a Uniform Magnetic Field

If a particle of mass m moves in a circle of radius r at a constant speed v , what acts on the particle is a radial force of magnitude $F = mv^2/r$ that always points toward the center and is perpendicular to the velocity of the particle.

In Section 8.2, we have also shown that the magnetic force \vec{F}_B always points in the direction perpendicular to the velocity \vec{v} of the charged particle and the magnetic field \vec{B} . Since \vec{F}_B can do no work, it can only change the direction of \vec{v} but not its magnitude. What would happen if a charged particle moves through a uniform magnetic field \vec{B} with its initial velocity \vec{v} at a right angle to \vec{B} ? For simplicity, let the charge be $+q$ and the direction of \vec{B} be into the page. It turns out that \vec{F}_B will play the role of a centripetal force and the charged particle will move in a circular path in a counterclockwise direction, as shown in Figure 8.5.1.

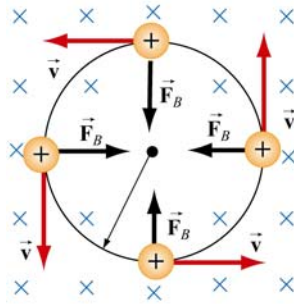


Figure 8.5.1 Path of a charge particle moving in a uniform \vec{B} field with velocity \vec{v} initially perpendicular to \vec{B} .

With

$$qvB = \frac{mv^2}{r} \quad (8.5.1)$$

the radius of the circle is found to be

$$r = \frac{mv}{qB} \quad (8.5.2)$$

The period T (time required for one complete revolution) is given by

$$T = \frac{2\pi r}{v} = \frac{2\pi}{v} \frac{mv}{qB} = \frac{2\pi m}{qB} \quad (8.5.3)$$

Similarly, the angular speed (cyclotron frequency) ω of the particle can be obtained as

$$\omega = 2\pi f = \frac{v}{r} = \frac{qB}{m} \quad (8.5.4)$$

If the initial velocity of the charged particle has a component parallel to the magnetic field \vec{B} , instead of a circle, the resulting trajectory will be a helical path, as shown in Figure 8.5.2:

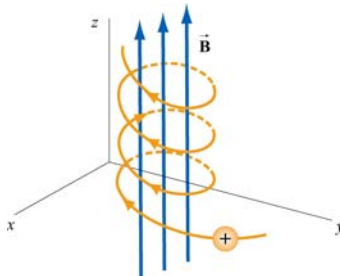


Figure 8.5.2 Helical path of a charged particle in an external magnetic field. The velocity of the particle has a non-zero component along the direction of \vec{B} .

8.5.1 Charged Particle Moving in a Uniform Magnetic Field Animation

Figure 8.5.3 shows a charge moving toward a region where the magnetic field is vertically upward. When the charge enters the region where the external magnetic field is non-zero, it is deflected in a direction perpendicular to that field and to its velocity as it enters the field. This causes the charge to move in an arc that is a segment of a circle, until the charge exits the region where the external magnetic field is non-zero. We show in the animation the total magnetic field which is the sum of the external magnetic field and the magnetic field of the moving charge (to be shown in Chapter 9):

$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi} \frac{q\vec{\mathbf{v}} \times \hat{\mathbf{r}}}{r^2} \quad (8.5.5)$$

The bulging of that field on the side opposite the direction in which the particle is pushed is due to the buildup in magnetic pressure on that side. It is this pressure that causes the charge to move in a circle.

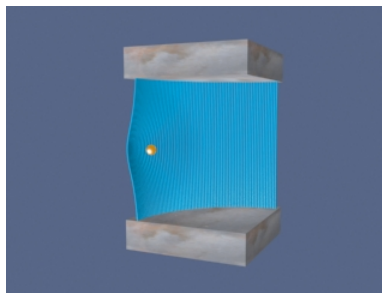


Figure 8.5.3 A charged particle moves in a magnetic field that is non-zero over the pie-shaped region shown. The external field is upward ([link](#))

Finally, consider momentum conservation. The moving charge in the animation of Figure 8.5.3 changes its direction of motion by ninety degrees over the course of the animation. How do we conserve momentum in this process? Momentum is conserved because momentum is transmitted by the field *from* the moving charge *to* the currents that are generating the constant external field. This is plausible given the field configuration shown in Figure 8.5.3. The magnetic field stress, which *pushes* the moving charge sideways, is accompanied by a tension *pulling* the current source in the opposite direction. To see this, look closely at the field stresses where the external field lines enter the region where the currents that produce them are hidden, and remember that the magnetic field acts as if it were exerting a tension parallel to itself. The momentum loss by the moving charge is transmitted to the hidden currents producing the constant field in this manner.

8.6 Applications

There are many applications involving charged particles moving through a uniform magnetic field.

8.6.1 Velocity Selector

In the presence of both electric field \vec{E} and magnetic field \vec{B} , the total force on a charged particle is

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \quad (8.6.1)$$

This is known as the Lorentz force. By combining the two fields, particles which move with a certain velocity can be selected. This was the principle used by J. J. Thomson to measure the charge-to-mass ratio of the electrons. In Figure 8.6.1 the schematic diagram of Thomson's apparatus is depicted.

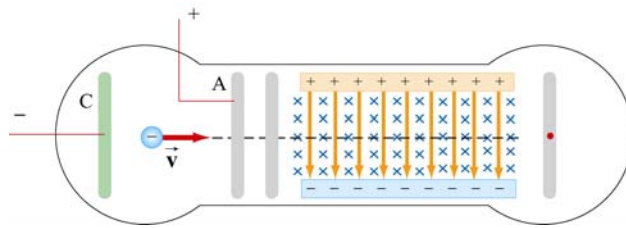


Figure 8.6.1 Thomson's apparatus

The electrons with charge $q = -e$ and mass m are emitted from the cathode C and then accelerated toward slit A. Let the potential difference between A and C be $V_A - V_C = \Delta V$. The change in potential energy is equal to the external work done in accelerating the electrons: $\Delta U = W_{\text{ext}} = q\Delta V = -e\Delta V$. By energy conservation, the kinetic energy gained is $\Delta K = -\Delta U = mv^2/2$. Thus, the speed of the electrons is given by

$$v = \sqrt{\frac{2e\Delta V}{m}} \quad (8.6.2)$$

If the electrons further pass through a region where there exists a downward uniform electric field, the electrons, being negatively charged, will be deflected upward. However, if in addition to the electric field, a magnetic field directed into the page is also applied, then the electrons will experience an additional downward magnetic force $-e\vec{v} \times \vec{B}$. When the two forces exactly cancel, the electrons will move in a straight path. From Eq. 8.6.1, we see that when the condition for the cancellation of the two forces is given by $eE = evB$, which implies

$$v = \frac{E}{B} \quad (8.6.3)$$

In other words, only those particles with speed $v = E/B$ will be able to move in a straight line. Combining the two equations, we obtain

$$\frac{e}{m} = \frac{E^2}{2(\Delta V)B^2} \quad (8.6.4)$$

By measuring E , ΔV and B , the charge-to-mass ratio can be readily determined. The most precise measurement to date is $e/m = 1.758820174(71) \times 10^{11}$ C/kg.

8.6.2 Mass Spectrometer

Various methods can be used to measure the mass of an atom. One possibility is through the use of a mass spectrometer. The basic feature of a *Bainbridge* mass spectrometer is illustrated in Figure 8.6.2. A particle carrying a charge $+q$ is first sent through a velocity selector.

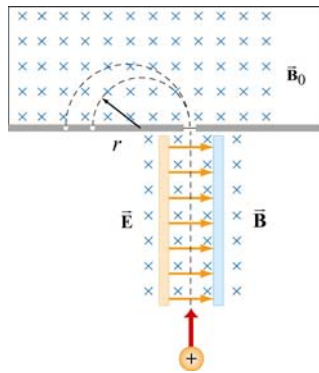


Figure 8.6.2 A Bainbridge mass spectrometer

The applied electric and magnetic fields satisfy the relation $E = vB$ so that the trajectory of the particle is a straight line. Upon entering a region where a second magnetic field \vec{B}_0 pointing into the page has been applied, the particle will move in a circular path with radius r and eventually strike the photographic plate. Using Eq. 8.5.2, we have

$$r = \frac{mv}{qB_0} \quad (8.6.5)$$

Since $v = E/B$, the mass of the particle can be written as

$$m = \frac{qB_0 r}{v} = \frac{qB_0 B r}{E} \quad (8.6.6)$$

8.7 Summary

- The **magnetic force** acting on a charge q traveling at a velocity \vec{v} in a magnetic field \vec{B} is given by

$$\vec{F}_B = q\vec{v} \times \vec{B}$$

- The magnetic force acting on a wire of length $\vec{\ell}$ carrying a steady current I in a magnetic field \vec{B} is

$$\vec{F}_B = I\vec{\ell} \times \vec{B}$$

- The magnetic force $d\vec{F}_B$ generated by a small portion of current I of length $d\vec{s}$ in a magnetic field \vec{B} is

$$d\vec{F}_B = I d\vec{s} \times \vec{B}$$

- The **torque** $\vec{\tau}$ acting on a close loop of wire of area A carrying a current I in a uniform magnetic field \vec{B} is

$$\vec{\tau} = I\vec{A} \times \vec{B}$$

where \vec{A} is a vector which has a magnitude of A and a direction perpendicular to the loop.

- The **magnetic dipole moment** of a closed loop of wire of area A carrying a current I is given by

$$\vec{\mu} = I\vec{A}$$

- The torque exerted on a magnetic dipole $\vec{\mu}$ placed in an external magnetic field \vec{B} is

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

- The potential energy of a magnetic dipole placed in a magnetic field is

$$U = -\vec{\mu} \cdot \vec{B}$$

- If a particle of charge q and mass m enters a magnetic field of magnitude B with a velocity \vec{v} perpendicular to the magnetic field lines, the radius of the circular path that the particle follows is given by

$$r = \frac{mv}{|q|B}$$

and the angular speed of the particle is

$$\omega = \frac{|q|B}{m}$$

8.8 Problem-Solving Tips

In this Chapter, we have shown that in the presence of both magnetic field \vec{B} and the electric field \vec{E} , the total force acting on a moving particle with charge q is $\vec{F} = \vec{F}_e + \vec{F}_B = q(\vec{E} + \vec{v} \times \vec{B})$, where \vec{v} is the velocity of the particle. The direction of \vec{F}_B involves the cross product of \vec{v} and \vec{B} , based on the right-hand rule. In Cartesian coordinates, the unit vectors are \hat{i} , \hat{j} and \hat{k} which satisfy the following properties:

$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k}, \quad \hat{k} \times \hat{j} = -\hat{i}, \quad \hat{i} \times \hat{k} = -\hat{j}$$

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

For $\vec{v} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k}$ and $\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$, the cross product may be obtained as

$$\vec{v} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ B_x & B_y & B_z \end{vmatrix} = (v_y B_z - v_z B_y) \hat{i} + (v_z B_x - v_x B_z) \hat{j} + (v_x B_y - v_y B_x) \hat{k}$$

If only the magnetic field is present, and \vec{v} is perpendicular to \vec{B} , then the trajectory is a circle with a radius $r = mv / |q|B$, and an angular speed $\omega = |q|B / m$.

When dealing with a more complicated case, it is useful to work with individual force components. For example,

$$F_x = ma_x = qE_x + q(v_y B_z - v_z B_y)$$

8.9 Solved Problems

8.9.1 Rolling Rod

A rod with a mass m and a radius R is mounted on two parallel rails of length a separated by a distance ℓ , as shown in the Figure 8.9.1. The rod carries a current I and rolls without slipping along the rails which are placed in a uniform magnetic field $\vec{\mathbf{B}}$ directed into the page. If the rod is initially at rest, what is its speed as it leaves the rails?

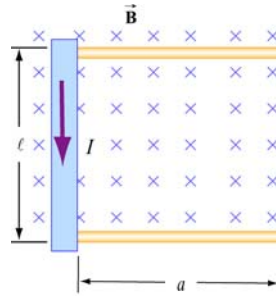
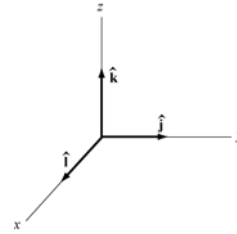


Figure 8.9.1 Rolling rod in uniform magnetic field

Solution:

Using the coordinate system shown on the right, the magnetic force acting on the rod is given by

$$\vec{\mathbf{F}}_B = I\vec{\ell} \times \vec{\mathbf{B}} = I(\ell \hat{\mathbf{i}}) \times (-B \hat{\mathbf{k}}) = I\ell B \hat{\mathbf{j}} \quad (8.9.1)$$



The total work done by the magnetic force on the rod as it moves through the region is

$$W = \int \vec{\mathbf{F}}_B \cdot d\vec{\mathbf{s}} = F_B a = (I\ell B)a \quad (8.9.2)$$

By the work-energy theorem, W must be equal to the change in kinetic energy:

$$\Delta K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \quad (8.9.3)$$

where both translation and rolling are involved. Since the moment of inertia of the rod is given by $I = mR^2/2$, and the condition of rolling with slipping implies $\omega = v/R$, we have

$$I\ell Ba = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{mR^2}{2}\right)\left(\frac{v}{R}\right)^2 = \frac{1}{2}mv^2 + \frac{1}{4}mv^2 = \frac{3}{4}mv^2 \quad (8.9.4)$$

Thus, the speed of the rod as it leaves the rails is

$$v = \sqrt{\frac{4I\ell Ba}{3m}} \quad (8.9.5)$$

8.9.2 Suspended Conducting Rod

A conducting rod having a mass density λ kg/m is suspended by two flexible wires in a uniform magnetic field \vec{B} which points out of the page, as shown in Figure 8.9.2.

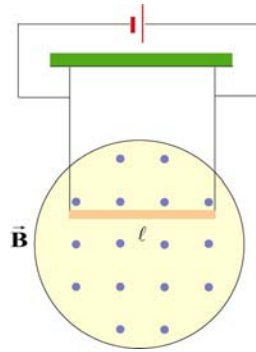
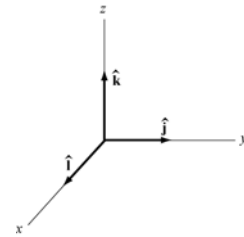


Figure 8.9.2 Suspended conducting rod in uniform magnetic field

If the tension on the wires is zero, what are the magnitude and the direction of the current in the rod?

Solution:

In order that the tension in the wires be zero, the magnetic force $\vec{F}_B = I\vec{\ell} \times \vec{B}$ acting on the conductor must exactly cancel the downward gravitational force $\vec{F}_g = -mg\hat{k}$.



For \vec{F}_B to point in the $+z$ -direction, we must have $\vec{\ell} = -\ell\hat{j}$, i.e., the current flows to the left, so that

$$\vec{F}_B = I\vec{\ell} \times \vec{B} = I(-\ell\hat{j}) \times (B\hat{i}) = -I\ell B(\hat{j} \times \hat{i}) = +I\ell B\hat{k} \quad (8.9.6)$$

The magnitude of the current can be obtain from

$$I\ell B = mg \quad (8.9.7)$$

or

$$I = \frac{mg}{B\ell} = \frac{\lambda g}{B} \quad (8.9.8)$$

8.9.3 Charged Particles in Magnetic Field

Particle A with charge q and mass m_A and particle B with charge $2q$ and mass m_B , are accelerated from rest by a potential difference ΔV , and subsequently deflected by a uniform magnetic field into semicircular paths. The radii of the trajectories by particle A and B are R and $2R$, respectively. The direction of the magnetic field is perpendicular to the velocity of the particle. What is their mass ratio?

Solution:

The kinetic energy gained by the charges is equal to

$$\frac{1}{2}mv^2 = q\Delta V \quad (8.9.9)$$

which yields

$$v = \sqrt{\frac{2q\Delta V}{m}} \quad (8.9.10)$$

The charges move in semicircles, since the magnetic force points radially inward and provides the source of the centripetal force:

$$\frac{mv^2}{r} = qvB \quad (8.9.11)$$

The radius of the circle can be readily obtained as:

$$r = \frac{mv}{qB} = \frac{m}{qB} \sqrt{\frac{2q\Delta V}{m}} = \frac{1}{B} \sqrt{\frac{2m\Delta V}{q}} \quad (8.9.12)$$

which shows that r is proportional to $(m/q)^{1/2}$. The mass ratio can then be obtained from

$$\frac{r_A}{r_B} = \frac{(m_A/q_A)^{1/2}}{(m_B/q_B)^{1/2}} \Rightarrow \frac{R}{2R} = \frac{(m_A/q)^{1/2}}{(m_B/2q)^{1/2}} \quad (8.9.13)$$

which gives

$$\frac{m_A}{m_B} = \frac{1}{8} \quad (8.9.14)$$

8.9.4 Bar Magnet in Non-Uniform Magnetic Field

A bar magnet with its north pole up is placed along the symmetric axis below a horizontal conducting ring carrying current I , as shown in the Figure 8.9.3. At the location of the ring, the magnetic field makes an angle θ with the vertical. What is the force on the ring?

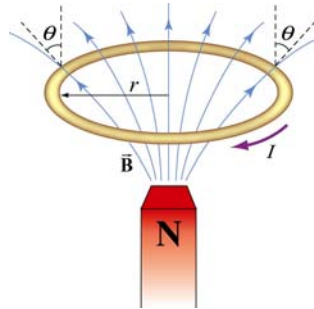


Figure 8.9.3 A bar magnet approaching a conducting ring

Solution:

The magnetic force acting on a small differential current-carrying element $I d\vec{s}$ on the ring is given by $d\vec{F}_B = I d\vec{s} \times \vec{B}$, where \vec{B} is the magnetic field due to the bar magnet. Using cylindrical coordinates $(\hat{r}, \hat{\phi}, \hat{z})$ as shown in Figure 8.9.4, we have

$$d\vec{F}_B = I(-ds \hat{\phi}) \times (B \sin \theta \hat{r} + B \cos \theta \hat{z}) = (IBds) \sin \theta \hat{z} - (IBds) \cos \theta \hat{r} \quad (8.9.15)$$

Due to the axial symmetry, the radial component of the force will exactly cancel, and we are left with the z -component.

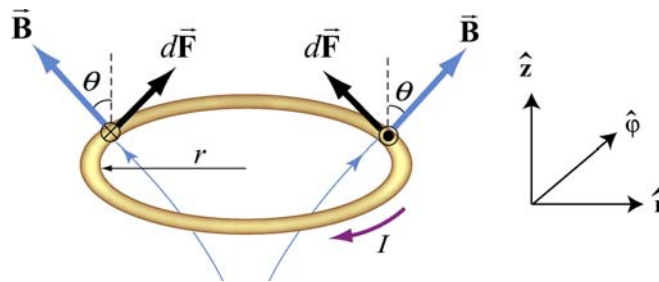


Figure 8.9.4 Magnetic force acting on the conducting ring

The total force acting on the ring then becomes

$$\vec{F}_B = (IB \sin \theta) \hat{z} \int ds = (2\pi r IB \sin \theta) \hat{z} \quad (8.9.16)$$

The force points in the $+z$ direction and therefore is repulsive.

8.10 Conceptual Questions

1. Can a charged particle move through a uniform magnetic field without experiencing any force? Explain.
2. If no work can be done on a charged particle by the magnetic field, how can the motion of the particle be influenced by the presence of a field?
3. Suppose a charged particle is moving under the influence of both electric and magnetic fields. How can the effect of the two fields on the motion of the particle be distinguished?
4. What type of magnetic field can exert a force on a magnetic dipole? Is the force repulsive or attractive?
5. If a compass needle is placed in a uniform magnetic field, is there a net magnetic force acting on the needle? Is there a net torque?

8.11 Additional Problems

8.11.1 Force Exerted by a Magnetic Field

The electrons in the beam of television tube have an energy of 12 keV ($1 \text{ eV} = 1.6 \times 10^{-19} \text{ J}$). The tube is oriented so that the electrons move horizontally from south to north. At MIT, the Earth's magnetic field points roughly vertically down (i.e. neglect the component that is directed toward magnetic north) and has magnitude $B \sim 5 \times 10^{-5} \text{ T}$.

- (a) In what direction will the beam deflect?
- (b) What is the acceleration of a given electron associated with this deflection? [Ans. $\sim 10^{-15} \text{ m/s}^2$.]
- (c) How far will the beam deflect in moving 0.20 m through the television tube?

8.11.2 Magnetic Force on a Current Carrying Wire

A square loop of wire, of length $\ell = 0.1 \text{ m}$ on each side, has a mass of 50 g and pivots about an axis AA' that corresponds to a horizontal side of the square, as shown in Figure 8.11.1. A magnetic field of 500 G, directed vertically downward, uniformly fills the region in the vicinity of the loop. The loop carries a current I so that it is in equilibrium at $\theta = 20^\circ$.

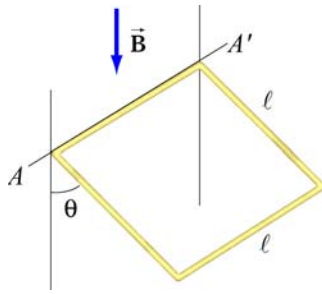


Figure 8.11.1 Magnetic force on a current-carrying square loop.

- Consider the force on each segment separately and find the direction of the current that flows in the loop to maintain the 20° angle.
- Calculate the torque about the axis due to these forces.
- Find the current in the loop by requiring the sum of all torques (about the axis) to be zero. (Hint: Consider the effect of gravity on each of the 4 segments of the wire separately.) [Ans. $I \sim 20$ A.]
- Determine the magnitude and direction of the force exerted on the axis by the pivots.
- Repeat part (b) by now using the definition of a magnetic dipole to calculate the torque exerted on such a loop due to the presence of a magnetic field.

8.11.3 Sliding Bar

A conducting bar of length ℓ is placed on a frictionless inclined plane which is tilted at an angle θ from the horizontal, as shown in Figure 8.11.2.

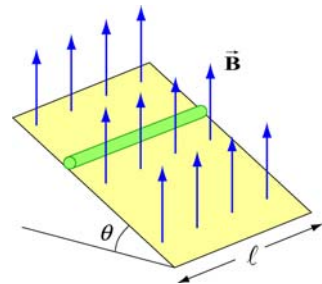


Figure 8.11.2 Magnetic force on a conducting bar

A uniform magnetic field is applied in the vertical direction. To prevent the bar from sliding down, a voltage source is connected to the ends of the bar with current flowing through. Determine the magnitude and the direction of the current such that the bar will remain stationary.

8.11.4 Particle Trajectory

A particle of charge $-q$ is moving with a velocity \vec{v} . It then enters midway between two plates where there exists a uniform magnetic field pointing into the page, as shown in Figure 8.11.3.

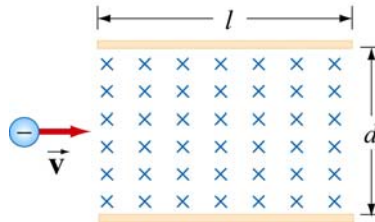


Figure 8.11.3 Charged particle moving under the influence of a magnetic field

- Is the trajectory of the particle deflected upward or downward?
- Compute the distance between the left end of the plate and where the particle strikes.

8.11.5 Particle Orbits in a Magnetic Field

Suppose the entire x - y plane to the right of the origin O is filled with a uniform magnetic field \vec{B} pointing out of the page, as shown in Figure 8.11.4.

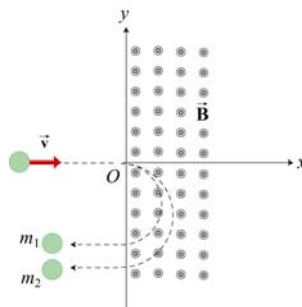


Figure 8.11.4

Two charged particles travel along the negative x axis in the positive x direction, each with speed v , and enter the magnetic field at the origin O . The two particles have the same charge q , but have different masses, m_1 and m_2 . When in the magnetic field, their trajectories both curve in the same direction, but describe semi-circles with different radii. The radius of the semi-circle traced out by particle 2 is exactly *twice* as big as the radius of the semi-circle traced out by particle 1.

- Is the charge q of these particles such that $q > 0$, or is $q < 0$?

(b) *Derive* (do not simply state) an expression for the radius R_1 of the semi-circle traced out by particle 1, in terms of q , v , B , and m_1 .

(c) What is the ratio m_2/m_1 ?

(d) Is it possible to apply an electric field \vec{E} in the region $x > 0$ only which will cause both particles to continue to move in a straight line after they enter the region $x > 0$? If so, indicate the magnitude and direction of that electric field, in terms of the quantities given. If not, why not?

8.11.6 Force and Torque on a Current Loop

A current loop consists of a semicircle of radius R and two straight segments of length ℓ with an angle θ between them. The loop is then placed in a uniform magnetic field pointing to the right, as shown in Figure 8.11.5.

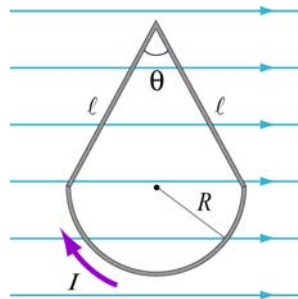


Figure 8.11.5 Current loop placed in a uniform magnetic field

(a) Find the net force on the current loop.

(b) Find the net torque on the current loop.

8.11.7 Force on a Wire

A straight wire of length 0.2 m carries a 7.0 A current. It is immersed in a uniform magnetic field of 0.1 T whose direction lies 20 degrees from the direction of the current.

(a) What is the direction of the force on the wire? Make a sketch to show your answer.

(b) What is the magnitude of the force? [Ans. ~0.05 N]

(c) How could you maximize the force without changing the field or current?

8.11.8 Levitating Wire

A copper wire of diameter d carries a current density \vec{J} at the Earth's equator where the Earth's magnetic field is horizontal, points north, and has magnitude $B = 0.5 \times 10^{-4} \text{ T}$. The wire lies in a plane that is parallel to the surface of the Earth and is oriented in the east-west direction. The density and resistivity of copper are $\rho_m = 8.9 \times 10^3 \text{ kg/m}^3$ and $\rho = 1.7 \times 10^{-8} \Omega \cdot \text{m}$, respectively.

- (a) How large must \vec{J} be, and which direction must it flow in order to levitate the wire? Use $g = 9.8 \text{ m/s}^2$
- (b) When the wire is floating how much power will be dissipated per cubic centimeter?

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