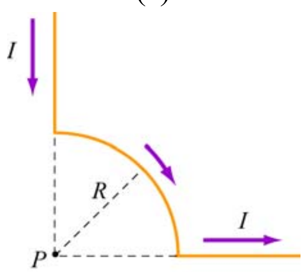
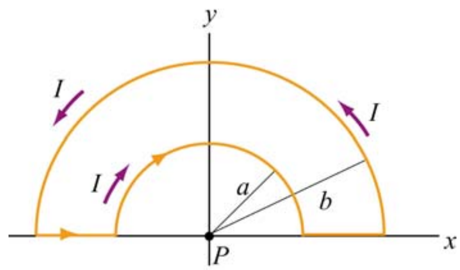
Creating Fields: Biot-Savart Law
Challenge Problem Solutions

Problem 1:
Find the magnetic field at point P due to the following current distributions:

(a)



(b)



Problem 1 Solution:
(a) The fields due to the straight wire segments are zero at P because $d\vec{\mathbf{s}}$ and $\hat{\mathbf{r}}$ are parallel or anti-parallel. For the field due to the arc segment, the magnitude of the magnetic field due to a differential current carrying element is given in this case by

$$
\begin{aligned}
d\vec{\mathbf{B}} &= \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}}\times\hat{\mathbf{r}}}{R^2} = \frac{\mu_0}{4\pi}\frac{IRd\theta(\sin\theta\,\hat{\mathbf{i}}-\cos\theta\,\hat{\mathbf{j}})\times(-\cos\theta\,\hat{\mathbf{i}}-\sin\theta\,\hat{\mathbf{j}})}{R^2} \\
&= -\frac{\mu_0}{4\pi}\frac{I(\sin^2\theta+\cos^2\theta)d\theta}{R}\hat{\mathbf{k}} = -\frac{\mu_0}{4\pi}\frac{Id\theta}{R}\hat{\mathbf{k}}
\end{aligned}.
$$

Therefore,

$$
\vec{\mathbf{B}} = -\int_0^{\pi/2}\frac{\mu_0 I}{4\pi R}d\theta\,\hat{\mathbf{k}} = -\frac{\mu_0 I}{4\pi R}\left(\frac{\pi}{2}\right)\hat{\mathbf{k}} = -\left(\frac{\mu_0 I}{8R}\right)\hat{\mathbf{k}} \text{ (or, into the page).}
$$

(b) There is no magnetic field due to the straight segments because point P is along the lines. Using the general expression for $d\vec{\mathbf{B}}$ obtained in (a), for the outer segment, we have

$$
\vec{\mathbf{B}}_{\text{out}} = \int_0^{\pi}\frac{\mu_0}{4\pi}\frac{Id\theta}{b}\hat{\mathbf{k}} = \left(\frac{\mu_0 I}{4b}\right)\hat{\mathbf{k}}
$$

Similarly, the contribution to the magnetic field from the inner segment is

$$
\vec{\mathbf{B}}_{\text{in}} = \int_{\pi}^{0}\frac{\mu_0}{4\pi}\frac{Id\theta}{a}\hat{\mathbf{k}} = -\left(\frac{\mu_0 I}{4a}\right)\hat{\mathbf{k}}.
$$

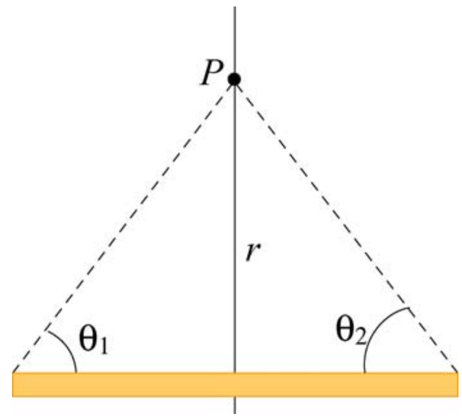
Therefore the net magnetic field at Point P is

$$\vec{\mathbf{B}}_{\text{net}} = \vec{\mathbf{B}}_{\text{out}} + \vec{\mathbf{B}}_{\text{in}} = -\frac{\mu_0 I}{4}\left(\frac{1}{a} - \frac{1}{b}\right)\hat{\mathbf{k}}$$ (into the page since $a < b$).

Problem 2:

A conductor in the shape of a square loop of edge length $\ell = 0.400$ m carries a current $I =$ 10.0 A as in the figure.

(a) Calculate the magnitude and direction of the magnetic field at the center of the square.



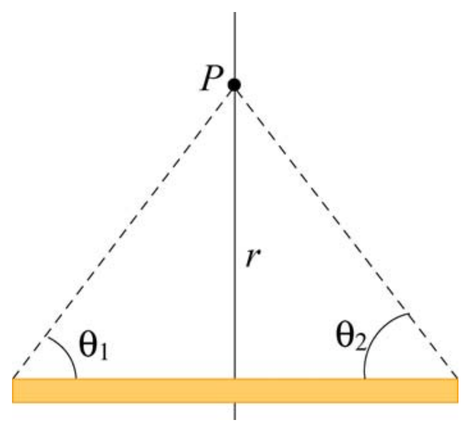
(b) If this conductor is formed into a single circular turn and carries the same current, what is the value of the magnetic field at the center?

Problem 2 Solutions:

For a finite wire carrying a current I, the contribution to the magnetic field at a point P is given by Eq. (9.1.5) of the Course Notes:

$$B = \frac{\mu_0 I}{4\pi r}(\cos\theta_1 + \cos\theta_2)$$

where θ_1 and θ_2 are the angles which parameterize the length of the wire.



Consider the bottom segment. The cosine of the angles are given by

$$\cos\theta_2 = \cos\theta_1 = \cos 45° = \frac{1}{\sqrt{2}}$$

This leads to

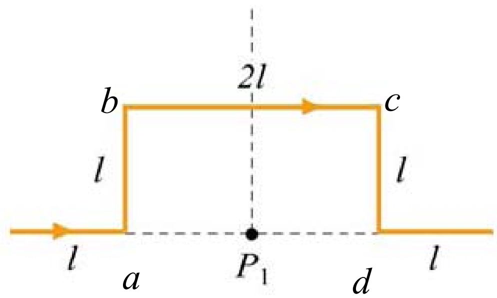
$$B_1 = \frac{\mu_0 I}{4\pi(l/2)}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = \frac{\mu_0 I}{\sqrt{2}\pi l}$$

The direction of $\vec{\mathbf{B}}_1$ is into the page. One may show that the other three segments yield the same contribution. Therefore, the total magnetic field at P is

$$B = 4B_1 = 2\sqrt{2}\,\frac{\mu_0 I}{\pi l} = 2\sqrt{2}\,\frac{(4\pi\times 10^{-7}\,\mathrm{T\cdot m/A})(10\,\mathrm{A})}{\pi(0.40\ \mathrm{m})} = 2.83\times 10^{-5}\ \mathrm{T}\ \text{(into the page)}$$

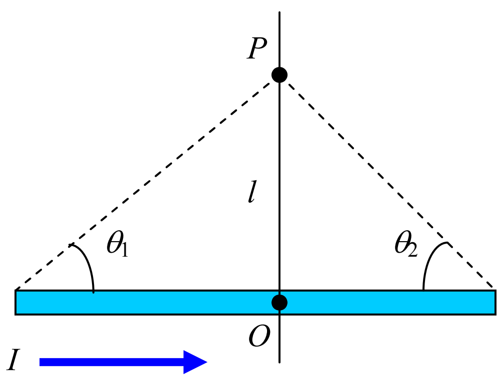
Problem 3:

A wire is bent into the shape shown on the right, and the magnetic field is measured at P_1 when the current in the wire is I.



From the discussion given in Example 9.1 The magnetic field is calculated as

$$B = \frac{\mu_0 I}{4\pi l}(\cos\theta_2 + \cos\theta_1)$$



For $a \to b$, $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = \frac{\pi}{4}$

$$B_{ab} = \frac{\mu_0 I}{4\pi l}\left(\frac{1}{\sqrt{2}} + 0\right) = \frac{\sqrt{2}\mu_0 I}{8\pi R}$$

For $b \to c$, $\theta_1 = \frac{\pi}{4}$ and $\theta_2 = \frac{\pi}{4}$

$$B_{bc} = \frac{\mu_0 I}{4\pi l}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2}\mu_0 I}{4\pi R}$$

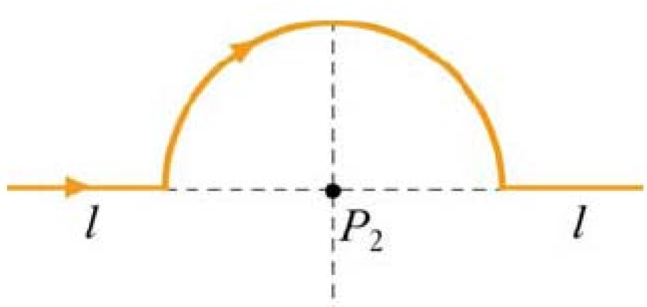
For $c \to d$, $\theta_1 = \frac{\pi}{4}$ and $\theta_2 = \frac{\pi}{2}$

$$B_{cd} = \frac{\mu_0 I}{4\pi l}\left(0 + \frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2}\mu_0 I}{8\pi R}$$

Therefore,

$$B_1 = B_{ab} + B_{bc} + B_{cd} = \frac{\sqrt{2}\mu_0 I}{2\pi l} \text{ (into page)}$$

The same segment of wire is then bent into a semi-circular shape shown in the figure below, and the magnetic field is measured at point P_2 when the current is again I. If the total length of wire is the same in each case, what is the ratio of B_1/B_2?



Problem 3 Solution:

$$\pi R = 4l \;\text{ or }\; R = \frac{4l}{\pi}$$

According to the Biot-Satart law, the magnitude of the magnetic field due to a differential current carrying element is given by

$$dB = \frac{\mu_0 I}{4\pi}\frac{|d\vec{s} \times \hat{r}|}{r^2} = \frac{\mu_0 I}{4\pi}\frac{R d\theta}{R^2} = \frac{\mu_0 I}{4\pi R} d\theta$$

Therefore,

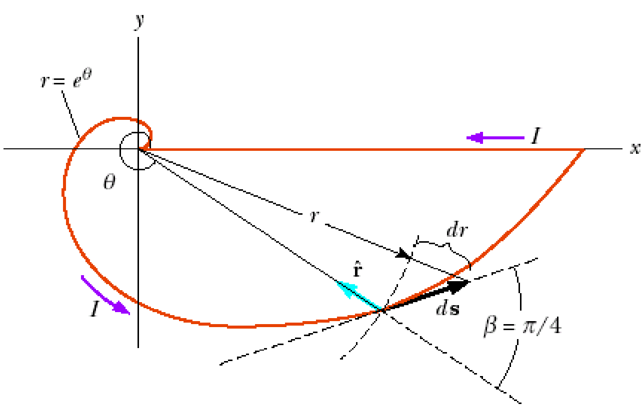
$$B_2 = \int_0^{\pi} \frac{\mu_0 I}{4\pi R} d\theta = \frac{\mu_0 I}{4\pi R}(\pi) = \frac{\mu_0 I}{4R} = \frac{\pi \mu_0 I}{16 l} \;\text{ (into page)}$$

Hence,

$$\frac{B_1}{B_2} = \left(\frac{\sqrt{2}\mu_0 I}{2\pi l}\right) \Big/ \left(\frac{\pi \mu_0 I}{16 l}\right) = \frac{16}{\sqrt{2}\pi^2} \approx 1.15$$

Problem 4:

A wire carrying a current I is bent into the shape of an exponential spiral, $r = e^{\theta}$, from $\theta = 0$ to $\theta = 2\pi$ as shown in the figure below.



To complete a loop, the ends of the spiral are connected by a straight wire along the x axis. Find the magnitude and direction of $\vec{\mathbf{B}}$ at the origin.

Hint: Use the Biot–Savart law. The angle β between a radial line and its tangent line at any point on the curve $r = f(\theta)$ is related to the function in the following way:

$$\tan\beta = \frac{r}{dr/d\theta}$$

Thus in this case $r = e^{\theta}$, $\tan\beta = 1$ and $\beta = \pi/4$. Therefore, the angle between $d\vec{\mathbf{s}}$ and $\hat{\mathbf{r}}$ is $\pi - \beta = 3\pi/4$. Also

$$ds = \frac{dr}{\sin(\pi/4)} = \sqrt{2}\, dr$$

Problem 4 Solution:

There is no contribution from the straight portion of the wire since $d\vec{\mathbf{s}} \times \hat{\mathbf{r}} = 0$. For the field of the spiral, we apply Biot-Savart law:

$$d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi}\frac{ds\sin\theta}{r^2}\hat{\mathbf{k}} = \frac{\mu_0 I}{4\pi}\frac{(\sqrt{2}\,dr)\sin(3\pi/4)}{r^2}\hat{\mathbf{k}}$$
$$= \frac{\mu_0 I}{4\pi}\frac{(\sqrt{2}\,dr)(1/\sqrt{2})}{r^2}\hat{\mathbf{k}} = \frac{\mu_0 I}{4\pi}\frac{dr}{r^2}\hat{\mathbf{k}}$$

Substituting $r = e^{\theta}$ and $dr = e^{\theta} d\theta$, the above expression becomes

$$d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi}\frac{e^{\theta} d\theta}{e^{2\theta}}\hat{\mathbf{k}} = \frac{\mu_0 I}{4\pi} e^{-\theta} d\theta\,\hat{\mathbf{k}}$$

Integrating the angle from 0 to 2π, we obtain

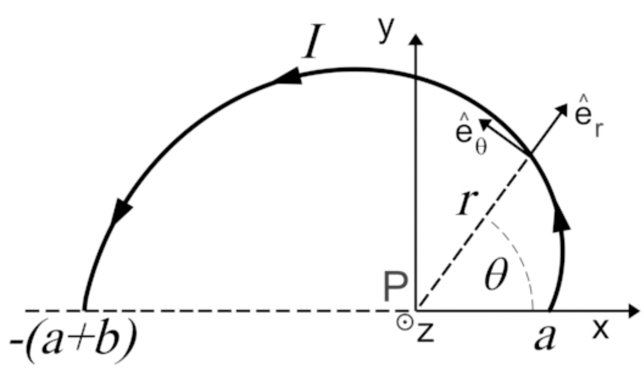
$$\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi}\hat{\mathbf{k}}\int_0^{2\pi} e^{-\theta} d\theta = \frac{\mu_0 I}{4\pi}\left(1 - e^{-2\pi}\right)\hat{\mathbf{k}}$$

Problem 5:

A wire segment is bent into the shape of an Archimedes spiral (see sketch). The equation that describes the curve in the range $0 \le \theta \le \pi$ is

$$r(\theta) = a + \frac{b}{\pi}\theta, \text{ for } 0 \le \theta \le \pi$$

where θ is the angle from the x-axis in radians. The point P is located at the origin of our xy coordinate system. The vectors $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta$ are the unit vectors in the radial and azimuthal directions, respectively, as shown. The wire segment carries current I, flowing in the sense indicated.



What is the magnetic field at point P?

Problem 5 Solution:

We should begin by calculating the magnetic field due to a small current segment ds (for example, at the location of the unit vectors on the above diagram). Using Biot Savart this creates a magnetic field:

$$\mathbf{d\vec{B}} = \frac{\mu_o I}{4\pi}\frac{\mathbf{d\vec{s}}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_o I}{4\pi}\frac{1}{r^2}\left(dr\,\hat{\mathbf{e}}_r + r\,d\theta\,\hat{\mathbf{e}}_\theta\right)\times\left(-\hat{\mathbf{e}}_r\right) = \frac{\mu_o I}{4\pi}\frac{r\,d\theta}{r^2}\hat{\mathbf{e}}_r\times\hat{\mathbf{e}}_\theta = \frac{\mu_o I}{4\pi}\frac{r\,d\theta}{r^2}\hat{\mathbf{k}}$$

$$\mathbf{d\vec{B}} = \frac{\mu_o I}{4\pi}\frac{d\theta}{r}\hat{\mathbf{k}}$$

Now we just need to plug in $r(\theta)$ and integrate in order to find the field due to the entire spiral.

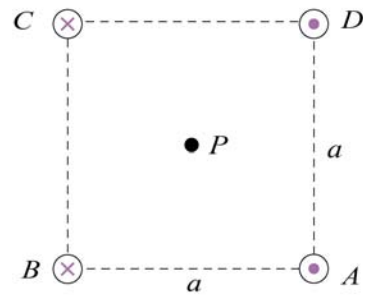
$$\vec{\mathbf{B}} = \hat{\mathbf{k}}\int_0^\pi \frac{\mu_o I}{4\pi}\frac{d\theta}{r} = \hat{\mathbf{k}}\int_0^\pi \frac{\mu_o I}{4\pi}\frac{d\theta}{\left(a+\frac{b}{\pi}\theta\right)} = \hat{\mathbf{k}}\frac{\mu_o I}{4\pi}\frac{\pi}{b}\ln\left(a+\frac{b}{\pi}\theta\right)\Bigg|_0^\pi = \boxed{\hat{\mathbf{k}}\frac{\mu_o I}{4}\frac{1}{b}\ln\left(1+\frac{b}{a}\right)}$$

where we made a simplification: $\ln\left(a+b\right)-\ln\left(a\right) = \ln\left(\frac{a+b}{a}\right) = \ln\left(1+\frac{b}{a}\right)$

Of course, you should always do a reality check. In the limit that b is small, we can use the approximation $\ln(1+\frac{b}{a}) \cong \frac{b}{a}$ and our expression becomes $\vec{\mathbf{B}} = \hat{\mathbf{k}}\frac{\mu_o I}{4}\frac{1}{b}\frac{b}{a} = \hat{\mathbf{k}}\frac{\mu_o I}{4a}$. This is what we expect because this is half the field at the center of a circle, and in the limit that *b* goes to zero our spiral becomes a semi-circle.

Problem 6:

Four infinitely long parallel wires carrying equal current I are arranged in such a way that when looking at the cross section, they are at the corners of a square, as shown in the figure below. Currents in A and D point out of the page, and into the page at B and C. What is the magnetic field at the center of the square?



Problem 6 Solution:

Four infinitely long parallel wires carrying equal current I are arranged in such a way that when looking at the cross section, they are at the corners of a square, as shown in the figure below. Currents in A and D point out of the page, and into the page at B and C. What is the magnetic field at the center of the square?

The magnitude of the magnetic field a distance r from an infinite wire is

$$B = \frac{\mu_0 I}{2\pi r}$$

The direction of the field is azimuthal in a sense given by using the right hand rule. Thus, the magnetic field due to each wire at point P is

$$\vec{\mathbf{B}}_A = B\hat{\mathbf{r}}_A = \frac{\mu_0 I}{2\pi(a/\sqrt{2})}\left(-\frac{1}{\sqrt{2}}\hat{\mathbf{i}} - \frac{1}{\sqrt{2}}\hat{\mathbf{j}}\right)$$
$$\vec{\mathbf{B}}_B = B\hat{\mathbf{r}}_B = \frac{\mu_0 I}{2\pi(a/\sqrt{2})}\left(\frac{1}{\sqrt{2}}\hat{\mathbf{i}} - \frac{1}{\sqrt{2}}\hat{\mathbf{j}}\right)$$
$$\vec{\mathbf{B}}_C = B\hat{\mathbf{r}}_C = \frac{\mu_0 I}{2\pi(a/\sqrt{2})}\left(-\frac{1}{\sqrt{2}}\hat{\mathbf{i}} - \frac{1}{\sqrt{2}}\hat{\mathbf{j}}\right)$$
$$\vec{\mathbf{B}}_D = B\hat{\mathbf{r}}_D = \frac{\mu_0 I}{2\pi(a/\sqrt{2})}\left(\frac{1}{\sqrt{2}}\hat{\mathbf{i}} - \frac{1}{\sqrt{2}}\hat{\mathbf{j}}\right)$$

Adding up the individual contributions, we have

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_A + \vec{\mathbf{B}}_B + \vec{\mathbf{B}}_C + \vec{\mathbf{B}}_D = -\frac{2\mu_0 I}{\pi a}\hat{\mathbf{j}}$$