Exercise 1 (20 points) Consider the following specifications of a clock.

Check which one of the following equalities hold and explain the reason.

1. \( S \) and \( T \) are strongly bisimilar,
2. \( S \) and \( U \) are branching bisimilar,
3. \( S \) and \( V \) are branching bisimilar,
4. \( S \) and \( V \) are rooted branching bisimilar.

Answer 1

1. \( S \) and \( T \) are bisimilar because there exists a bisimulation relation \( R = \{ (s_0, t_0), (s_1, t_1), (s_1, t_2), (s_0, t_3) \} \) such that \( (s_0, t_0) \in R \).
2. \( S \) and \( U \) are branching bisimilar because there exists a branching bisimulation relation \( R = \{ (s_0, t_0), (s_1, t_1) \} \) such that \( (s_0, t_0) \in R \).
3. \( S \) and \( V \) are not branching bisimilar. Assume that they were branching bisimilar, then there would exist a branching bisimulation relation \( R \) such that \( (s_0, v_0) \in R \). Since \( v_0 \xrightarrow{\tau} v_1 \), the only possibly to mimic this behavior in \( S \) is to remain in \( s_0 \), which means that \( (s_0, v_1) \in R \). However, \( v_1 \xrightarrow{\text{time}} v_0 \), but this transition cannot be mimicked by \( s_0 \).
4. \( S \) and \( V \) are not rooted branching bisimilar, because they are not branching bisimilar.
Exercise 2 (20 points) Assume that the sort $i\text{Natural}$ of natural numbers is defined as follows:

- **sort** $i\text{Natural}$;
- **cons** zero: $i\text{Natural}$;
  succ: $i\text{Natural} \rightarrow i\text{Natural}$;
- **map** eq: $i\text{Natural} \times i\text{Natural} \rightarrow \text{Bool}$;
- **var** $i, j$: $i\text{Natural}$;
- **eqn** eq(i, i)= true; (1)
  eq(zero, succ(i))= false; (2)
  eq(succ(i), zero)= false; (3)
  eq(succ(i), succ(j))= eq(i,j); (4)

1. Prove that zero cannot be the same as succ(zero).

2. Define the operation multiply, which multiplies two natural numbers.

Answer 2

1. Assume towards a contradiction that zero is the same as succ(zero). Then the following derivation (leading to contradiction) follows.

$$
\begin{align*}
\text{true} & = & (1) \\
\text{eq(zero, zero)} & = & \text{(assumption)} \\
\text{eq(zero, succ(zero))} & = & (2) \\
\text{false} & = & \\
\end{align*}
$$

2.

- **var** $i, j$: Nat;
- **map** add: $i\text{Natural} \times i\text{Natural} \rightarrow i\text{Natural}$;
  multiply: $i\text{Natural} \times i\text{Natural} \rightarrow i\text{Natural}$;
- **eqn** add(zero, i)= i;
  add(succ(i), j)= succ(add(i, j));
  multiply(zero, i)= zero;
  multiply(succ(i), j)= add(i, multiply(i, j));
Exercise 3 (20 points) Prove the following equations using the axioms provided in the appendix.

1. \( c \rightarrow (c' \rightarrow x \circ y) \circ y = c \land c' \rightarrow x \circ y \),
2. \((a + a) \cdot (a + b) + (b + \delta) \cdot (a + b) + b \cdot (a + b) = (a + b) \cdot (a + b)\),
3. \( \delta \parallel a = a \cdot \delta \),
4. \( a \parallel (b + c) = (b + c) \cdot a + ((b + c) | a) + a \cdot (b + c) \).

Note that sequential composition binds stronger than nondeterministic choice.

Answer 3 1. By induction on \( c \).

2. \[
\begin{align*}
(a + a) \cdot (a + b) + (b + \delta) \cdot (a + b) + b \cdot (a + b) &= (A6) \\
(a + a) \cdot (a + b) + b \cdot (a + b) + b \cdot (a + b) &= (A3) \\
a \cdot (a + b) + b \cdot (a + b) + b \cdot (a + b) &= (A3) \\
(a + b) \cdot (a + b) &= (A4)
\end{align*}
\]

3. In the following derivations, \( \parallel \) and \( | \) both bind more strongly than + and less strongly than \( \cdot \).

\[
\begin{align*}
\delta \parallel a &= (M) \\
\delta \parallel a + a \parallel \delta + \delta | a &= (LM2) \\
\delta + a \parallel \delta + \delta | a &= (A1,A6) \\
a \parallel \delta + \delta | a &= (LM1) \\
a.\delta + \delta | a &= (S1,S4) \\
a.\delta &= (A6)
\end{align*}
\]

4. \[
\begin{align*}
a \parallel (b + c) &= (M) \\
a \parallel (b + c) + (b + c) \parallel a + a | (b + c) &= (LM1) \\
a \cdot (b + c) + (b + c) \parallel a + a | (b + c) &= (LM1) \times 2 \\
a \cdot (b + c) + b \cdot a + c \cdot a + a | (b + c) &= (LM1) \times 2 \\
a \cdot (b + c) + b \cdot a + c \cdot a + a | (b + c) &= (A4) \\
a \cdot (b + c) + (b + c) \cdot a + a | (b + c) &= (A1,S1) \\
a \cdot (b + c) + (b + c) | a + (b + c) \cdot a &= (LM4)
\end{align*}
\]

Exercise 4 (20 points) Give an mCRL2 specification for a simple ice-cream machine which can be refilled by executing action refill, when empty. After each refill, it can produce 100 ice-creams by executing action ice. At each point of time, it can also show its capacity (the number of ice-creams it can produce before refilling), by executing action togo(n), where \( n \) is a natural number denoting the capacity. The ice-cream machine is assumed to be initially empty.

Answer 4

```plaintext
act togo : Nat;
  refill, ice ;

proc IceMachine (cap: Nat) =
togo(cap) \cdot IceMachine(cap) +
(cap > 0) \rightarrow ice \cdot IceMachine(Int2Nat(cap-1)) +
(cap \approx 0) \rightarrow refill.IceMachine(100)
init IceMachine (0) ;
```
Exercise 5 (20 points) Specify the following properties in the Modal $\mu$-Calculus. Assume that the set of actions is $Act = \{fill, produce, empty\}$.

1. Directly after every $fill$ actions, either a $produce$ or an $empty$ action must be taken.
2. After each $empty$ action, another $empty$ cannot be done.
3. Always after each $empty$ action, eventually a $fill$ action will be taken.
4. There is no infinite path of only $produce$ actions.

Properties 1 and 2 should hold in the initial state and need not hold everywhere. Properties 3 and 4 should hold everywhere.

Answer 5

1. $[\text{fill}]( (\langle produce \rangle \text{true} \lor \langle empty \rangle \text{true}) \land [\text{fill}] \text{false} ),$
2. $[\text{empty}][\text{empty}] \text{false},$
3. $\nu X.([Act]X \land [empty]Y)$
   $\mu Y.([\{empty, produce\}]Y \land \langle Act \rangle \text{true} ),$
4. $\nu X.([Act]X \land [produce]Y)$
   $\mu Y.([\{produce\}]Y,$
A1 \[ x + y = y + x \]
A2 \[ x + (y + z) = (x + y) + z \]
A3 \[ x + x = x \]
A4 \[ (x + y)z = xz + yz \]
A5 \[ (xy)z = x(yz) \]
A6 \[ x + \delta = x \]
A7 \[ \delta x = \delta \]

Cond1 \[ \text{true} \rightarrow x \circ y = x \]
Cond2 \[ \text{false} \rightarrow x \circ y = y \]

SUM1 \[ \sum_{d:D} x = x \]
SUM3 \[ \sum_{d:D} X(d) = X(e) + \sum_{d:D} X(d) \]
SUM4 \[ \sum_{d:D} (X(d) + Y(d)) = \sum_{d:D} X(d) + \sum_{d:D} Y(d) \]
SUM5 \[ \sum_{d:D} X(d) \cdot y = \sum_{d:D} X(d) \cdot y \]

Table 1: Axioms for the basic operators

Note that \( \alpha \) and \( \beta \) range over (multi)actions and \( x, y \) and \( z \) range over processes.
\[ M \quad x \parallel y = x \parallel y + y \parallel x + y \]

| LM1 | \( \alpha \parallel x = \alpha \cdot x \) |
| LM2 | \( \delta \parallel x = \delta \) |
| LM3 | \( \alpha \cdot x \parallel y = \alpha \cdot (x \parallel y) \) |
| LM4 | \( (x + y) \parallel z = x \parallel z + y \parallel z \) |
| LM5 | \( (\sum_{d:D} X(d)) \parallel y = \sum_{d:D} X(d) \parallel y \) |

| S1 | \( x|y = y|x \) |
| S2 | \( (x|y)|z = x|(y|z) \) |
| S3 | \( x|\tau = x \) |
| S4 | \( \alpha|\delta = \delta \) |
| S5 | \( (\alpha\cdot x)|\beta = \alpha|\beta\cdot x \) |
| S6 | \( (\alpha\cdot x)|(\beta\cdot y) = \alpha|\beta \cdot (x \parallel y) \) |
| S7 | \( (x + y)|z = x|z + y|z \) |
| S8 | \( (\sum_{d:D} X(d))|y = \sum_{d:D} X(d)|y \) |

| TC1 | \( (x \parallel y) \parallel z = x \parallel (y \parallel z) \) |
| TC2 | \( x \parallel \delta = x \cdot \delta \) |
| TC3 | \( (x|y) \parallel z = x|(y \parallel z) \) |

Table 2: Axioms for the parallel composition operators