Flight and Orbital Mechanics

Solutions
Exam 2104 - January 2011

Solutions to flight mechanics questions

Question 1a (5 points)
\[ V = M \cdot a = M \sqrt{\gamma RT} = 0.8 \cdot \sqrt{1.4 \cdot 287.05 \cdot 255.65} = 256.4 \text{ [m/s]} \]
\[ H_e = \frac{E}{W} = \frac{\frac{1}{2} m V^2}{W} + \frac{m g H}{W} = \frac{V^2}{2g} + H = \frac{256.4^2}{2 \cdot 9.80665} + 5000 = 8352 \text{ [m]} \]

Question 1b (10 points)
Calculate the radius from the airspeed and angular velocity

\[ R = \frac{V}{\Omega} = \frac{256.4}{\pi/60} = 4897 \text{ [m]} \]

Now use the equations of motion to calculate the corresponding load factor

\[ L \sin \phi = \frac{W V^2}{gR} \]
\[ L \cos \phi = W \]
\[ n = \frac{L}{W} \]
\[ \cos^2 \phi + \sin^2 \phi = 1 \]
\[ n \left( 1 - \frac{1}{n^2} \right) = \frac{V^2}{gR} = \frac{256.4^2}{9.80665 \cdot 4897} \Rightarrow n = 1.69 \]

The bank angle can be calculated from the load factor

\[ \cos \phi = \frac{1}{n} = \frac{1}{1.69} \Rightarrow \phi = 53.9 \text{ [deg]} \]

Question 2a (5 points)
See FBD and KD in lecture sheets

Equations of motion:
\[ \frac{W}{g} \frac{dV}{dt} = T - D - W \sin \gamma \]
\[ \frac{W}{g} \frac{dy}{dt} = L - W \cos \gamma \]

Kinematic equations:
\[
\frac{ds}{dt} = V \cos \gamma (\approx V)
\]
\[
\frac{dH}{dt} = V \sin \gamma
\]

**Question 2b (5 points)**

\[
\frac{dV}{dt} = g \left[ \frac{T - D}{W} - \sin \gamma \right]
\]
\[
\frac{dV}{ds} = g \left[ \frac{T - D}{W} - \sin \gamma \right]
\]

\[
V dV = g \frac{T - D}{W} ds - g \sin \gamma ds
\]

\[
\int_{V_{scf}} V dV = \frac{g}{W} \int_{V_{scf}} (T - D) ds - \int_{H_{scf}}^H dH
\]

\[
\frac{V_{scf}^2 - V_{LOF}^2}{2g_0} = \frac{T - D}{W} \Delta s - H_{scf}
\]

\[
\Delta s = \frac{2g_0}{T - D} \frac{V_{scf}^2 - V_{LOF}^2 + H_{scf}}{W}
\]

**Question 2c (5 points)**

\[
L = W \Rightarrow V_{min} = \sqrt{\frac{W}{S \rho C_{L, max}}} = \sqrt{\frac{3260000}{510 \cdot 1.225 \cdot 1.8}} = 76.14 \text{ [m/s]}
\]

\[
V_{scf} = 1.3 \cdot V_{min} = 1.3 \cdot 76.14 = 98.98 \text{ [m/s]}
\]

\[
V_{LOF} = 1.2 \cdot V_{min} = 1.2 \cdot 76.14 = 91.37 \text{ [m/s]}
\]

\[
H_{scf} = 10.7 \text{ [m]}
\]

\[
\Delta s = \frac{2g_0}{T - D} \frac{98.98^2 - 91.37^2 + 10.7}{W} = \frac{275665600}{3260000}
\]

\[
\bar{T} = 4 \cdot 165 = 660 \text{ [kN]}
\]

\[
C_{L, scf} = \frac{W}{S \rho V_{scf}^2} = \frac{3260000}{510 \cdot 1.225 \cdot 98.98^2} = 1.07
\]

\[
C_{L, LOF} = \frac{V_{scf}^2}{V_{LOF}^2} = \frac{1.3^2}{1.2^2} \Rightarrow C_{L, LOF} = 1.25
\]
\[ C_D = C_a + \frac{C^2}{\pi Ae} \]
\[ C_{D,scr} = 0.036 + \frac{1.07^2}{\pi \cdot 6.7 \cdot 0.7} = 0.114 \]
\[ C_{D,LOF} = 0.036 + \frac{1.25^2}{\pi \cdot 6.7 \cdot 0.7} = 0.142 \]
\[ D_{scr} = \frac{C_{D,scr}}{C_{L,scr}} W = \frac{0.114}{1.07} \times 3260000 = 347 \text{ [kN]} \]
\[ D_{LOF} = \frac{C_{D,LOF}}{C_{L,LOF}} W = \frac{0.142}{1.25} \times 3260000 = 370 \text{ [kN]} \]
\[ \bar{T} - \bar{D} = \frac{(T - D)_{scr} + (T - D)_{LOF}}{2} = \frac{660 - 347 + 660 - 370}{2} = 302 \text{ [kN]} \]
\[ \Delta s = \frac{275665600}{\bar{T} - \bar{D}} = \frac{275665600}{302000} = 913 \text{ [m]} \]

**Question 2d (5 points)**
\[ T = 3 \cdot 165 = 495 \text{ [kN]} \]
\[ \bar{T} - \bar{D} = 302 - 495 = 137 \text{ [kN]} \]
\[ \Delta s = \frac{275665600}{\bar{T} - \bar{D}} = \frac{275665600}{137000} = 2012 \text{ [m]} \]

**Question 2e (5 points)**
See lecture sheets

**Question 2f (5 points)**
\[ 0 = N + L - W \]
\[ \frac{W \, dv}{g \, dt} = T - D - D_g = T - D - \mu N = T - D - \mu (W - L) \]

**Question 2g (5 points)**
\[ \frac{W \, dv}{g \, dt} = T - D - \mu (W - L) \]
\[ \frac{dv \, ds}{ds \, dt} = \frac{g}{W} (T - D - \mu (W - L)) \]
\[ \int_{v_{aux}}^{v} v \, dv = \frac{g}{W} \int_{s_{aux}}^{s} (T - D - \mu (W - L)) \, ds \]
\[ -\frac{V_{LOF}^2}{2} = \frac{g}{W} (\mu L - \mu W - \bar{D}) \Delta s \]
\[ \bar{C} = C_{L,0} \frac{k}{2} \rho \left( \frac{V_{T,0}}{\sqrt{2}} \right)^2 S = 0.7 \cdot \frac{1}{2} \cdot 1.225 \left( \frac{91.37}{\sqrt{2}} \right)^2 510 = 913 \text{ [kN]} \]

\[ C_{D,0} = 0.036 + \frac{0.7^2}{\pi \cdot 6.7 \cdot 0.7} = 0.0693 \]

\[ \bar{D} = C_{D,0} \frac{k}{2} \rho \left( \frac{V_{T,0}}{\sqrt{2}} \right)^2 S = 0.0693 \cdot \frac{1}{2} \cdot 1.225 \left( \frac{91.37}{\sqrt{2}} \right)^2 510 = 90.4 \text{ [kN]} \]

\[-\frac{V_{T,0}^2}{2} = \frac{g}{W} (\mu \bar{L} - \mu \bar{W} - \bar{D}) \Delta s \]

\[-4174 = \frac{9.80665}{3260 \cdot 10^3} (0.4 \cdot 913 \cdot 10^3 - 0.4 \cdot 3260 \cdot 10^3 - 90.4 \cdot 10^3) \Delta s \]

\[ \Delta s = 1348 \text{ [m]} \]

(Hence, in this case, it is better to brake (1348 m) than to continue the take-off (2012 m))
Questions and answers exam ae2-104 dated January 27, 2011 (space part)

**Question 3:**
The gravity potential of the Earth is given by the following equation:

\[
U = -\frac{\mu}{r} \left[ 1 - \sum_{n=2}^{\infty} J_n \left( \frac{R}{r} \right)^n P_n(\sin \delta) + \sum_{n=2}^{\infty} \sum_{m=1}^{n} J_{n,m} \left( \frac{R}{r} \right)^n P_{n,m}(\sin \delta) \cos(m(\lambda - \lambda_{n,m})) \right]
\]

Here, \( P_n(\sin \delta) \) and \( P_{n,m}(\sin \delta) \) represent the Legendre polynomials and functions, respectively:

\[
P_n(x) = \frac{1}{(-2)^n n!} \frac{d^n}{dx^n} (1-x^2)^n
\]

\[
P_{n,m}(x) = (1-x^2)^{m/2} \frac{d^m P_n(x)}{dx^m}
\]

a) (2 points) Compute the orbit radius of a geostationary satellite.

b) (3 points) Give the general expression to derive the East-West acceleration from the potential formulation for the gravity field.

c) (6 points) Derive the general equation for the East-West acceleration due to the term \((2,2)\) for an arbitrary satellite (i.e., arbitrary \(r, \delta, \lambda\)).

d) (2 points) What is the equation for the East-West acceleration due to \(J_{2,2}\) for a geostationary satellite (expressed in numbers, for arbitrary longitude)?

e) (2 points) What are the locations of the equilibrium points?

**Data:** \( \mu_{\text{Earth}} = 398600.4415 \, \text{km}^3/\text{s}^2; \) \( R_E = 6378.137 \, \text{km}; \) \( T_E = 23^h56^m4^s; \) \( J_{2,2} = 1.816 \times 10^{-6}; \) \( \lambda_{2,2} = -14.9^\circ. \)

**Answers:**

a) \( a = 42164.14 \, \text{km} \)

b) \( \text{a}_{\text{ew}} = -1/(r \cos \delta) \, \partial U/\partial \lambda \)

c) \( P_{2,2}(x) = 3(1-x^2) \rightarrow \text{a}_{\text{ew}2,2} = -6 \mu r^4 J_{2,2} R_e^2 \cos \delta \sin(2(\lambda - \lambda_{2,2})). \)

d) substitution of numbers: \( \text{a}_{\text{ew}2,2} = -5.59 \times 10^8 \, \sin(2(\lambda + 14.9)) \, [\text{m/s}^2] \)

e) \( \lambda = -14.9^\circ \) modulo 90°
Question 4:
Consider a hypothetical planet X with mass $5 \times 10^{25}$ kg, orbiting the Sun in a circular orbit with radius 3 AU. The orbital plane coincides with the ecliptic (i.e., the orbital plane of the Earth).

a) (3 points) Make a sketch of the situation when the gravitational attraction of this planet X on satellites around the Earth is largest.

b) (3 points) Idem for the case when this would be smallest.

c) (4 points) Compute the maximum perturbing acceleration due to this planet X, acting on a geostationary satellite (radius orbit is 42200 km). Remember: $\mu = G \times M$. Data: $G = 6.673 \times 10^{-20}$ km$^3$/kg/s$^2$; 1 AU = 149.6 $10^6$ km.

Answers:
a) see sheet 30 of lecture hours 19+20 (v1.2)
b) see sheet30 of lecture hours 19+20 (v1.2)
c) effective acceleration = direct acceleration – acceleration on Earth -> $a_{\text{acc}}^\text{max} = 1.0516 \times 10^{-11}$ m/s$^2$

Question 5:
One of the main issues for designing a space mission is the occurrence of eclipses.

a) (1 points) What is the definition of an eclipse?

b) (3 points) An eclipse has consequences for at least 3 subsystems of the satellite. What are these, and discuss the consequences for each one briefly (about 2 lines each).

c) (3 points) What are the two conditions that determine whether an Earth satellite is in eclipse or not? Give the mathematical conditions in a sketch and discuss each one briefly.

d) (3 points) One of the conditions can translate into the so-called shadow function as given below. Discuss the meaning of the various elements in the equation, and discuss the use of this equation.

\[ S(\theta) = R^2_e (1 + e \cos \theta)^2 + p^2 (\alpha \cos \theta + \beta \sin \theta)^2 - p^2 \]

Answers:
a) situation in which satellite not directly illuminated by Sun
b) power system, thermal control, attitude control, operations instruments, optical tracking from Earth, ….

c) satellite in front of Earth, and component satellite position perpendicular to direction to Sun must be smaller than Earth radius. See sheets for sketch.

d) see sheets. Used to assess whether satellite is in eclipse or not.
**Question 6:**
Consider the various high-thrust options for a transfer from Earth to Jupiter: a minimum-energy Hohmann transfer, or a faster transfer (demanding more energy).

a) (4 points) Compute the main characteristics of the Hohmann transfer: semi-major axis, eccentricity and transfer time.

b) (2 points) Compute the excess velocity $V_\infty$ (when escaping from Earth).

c) (2 points) Now assume that the launcher is able to give the spacecraft a $V_\infty$ which is 10.5 km/s (so about 20% higher), still in the direction parallel to the heliocentric velocity of Earth itself. What would be the heliocentric velocity of the spacecraft when leaving Earth?

d) (2 points) Compute the semi-major axis and the eccentricity of the new transfer orbit.

If question (c) could not be answered by you, use a value of 40 km/s for this heliocentric velocity.

e) (2 points) Compute the value for the true anomaly $\theta$, when arriving at Jupiter.

f) (3 points) Given the relations $\tan(\frac{E}{2}) = \sqrt{\frac{1-e}{1+e}} \tan(\frac{\theta}{2})$, $M = E - e \sin(E)$ and $M = n(t - t_0)$, compute the value for the eccentric anomaly $E$, the mean anomaly $M$ and the travel time.

Data: $\mu_{\text{Sun}} = 1.3271 \times 10^{11}$ km$^3$/s$^2$; distance Earth-Sun = 1 AU; distance Jupiter-Sun = 5.2 AU; 1 AU = 149.6 $10^6$ km.

**Answers:**

a) $a = 3.1$ AU; $r_{\text{peri}} = r_{\text{Earth}} \Rightarrow e = 0.6774$; $T_H = 86 \times 10^6$ sec or 996.84 days or 2.73 yrs

b) $E_{\text{kin}} + E_{\text{pot}} = E_{\text{tot}}$ values for $r$ and a known $\Rightarrow V_{\text{sat}} = 38.575$ km/s; $V_{\text{earth}} = V_{\text{circ}}$ at 1 AU $\Rightarrow 29.784$ km/s; $V_\infty = 8.791$ km/s

c) Sum of $V_{\text{earth}}$ and $V_\infty \Rightarrow 40.284$ km/s

d) $E_{\text{kin}} + E_{\text{pot}} = E_{\text{tot}}$ values for $V$ and $r$ known $\Rightarrow a = 8.7657 \times 10^8$ km (5.859 AU); position at Earth is pericenter $\Rightarrow e = 0.8293$

e) $r = a(1-e^2)/(1+e \cos \theta)$; values for $r$, $a$ and $e$ known $\Rightarrow \theta = 141.40^\circ$

f) substitution in eqs: $\theta \Rightarrow E \Rightarrow M$ (all in rad!!) $\Rightarrow \Delta t = 0.436 \times 10^8$ sec = 504.86 days = 1.38 yrs