Flight and Orbital Mechanics
Lecture hours 11, 12 – Cruise

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Semester 1 - 2012
Content

- Introduction
- Optimum cruise profile
  - Optimal airspeed for given H, W
  - Effect of altitude
  - Effect of weight
  - Best flying strategy
- Analytic Range equations
- Story
- Weight breakdown
- Economics
- Summary
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Introduction

Typical cruise flight

- Takeoff weight
- Climb
- Cruise
- Descent
- Landing weight
- Total range (stage length)
- Range R
Introduction

Objective

- Range (distance)
- Endurance (Maximum time)
Introduction

Equations of motion

General 2D equations of motion and power equation

Unsteady curved symmetric flight

\[
T \cos \alpha_T - D - W \sin \gamma = \frac{W}{g} \frac{dV}{dt}
\]

\[
L - W \cos \gamma + T \sin \alpha_T = \frac{W}{g} V \frac{d\gamma}{dt}
\]

\[
\frac{P_a - P_r}{W} = RC + \frac{V}{g} \frac{dV}{dt}
\]

Cruise flight

Quasi steady \((dV/dt = 0)\), quasi-rectilinear, \((d\gamma/dt = 0)\)

Weight of the aircraft is **not constant**

Small flight path angle \(\rightarrow \cos \gamma = 1, \sin \gamma \neq 0\)

Assume that the thrust vector acts in the direction of flight \((\alpha_T = 0)\)
Introduction

Equations of motion

Equations of motion cruise flight

\[ 0 = \frac{g}{W} (T - D - W \sin \gamma) \]
\[ L = W \quad (\text{quasi rectilinear}) \]

Additional equation

\[ \frac{dW}{dt} = -F(\Gamma, V, H) \]

Kinematic equations

\[ \frac{ds}{dt} = V \cos \gamma \]
\[ \frac{dH}{dt} = V \sin \gamma \]
Introduction

Problem definition

Pilot can choose a certain airspeed and altitude: $H(t)$, $V(t)$

What is the best flight condition?

1. Optimal initial conditions ($V$, $H$ at initial weight)
2. Optimal flying strategy ($V$, $H$ at decreasing weight)
Introduction

Criteria for optimal flight

1. Maximum endurance $E$:
   Fuel flow $F_{\text{min}}$ at every point in time

2. Maximum range $R$:
   Specific range $(V/F)_{\text{max}}$ at every point in time

3. Given range, minimum fuel:
   Specific range $(V/F)_{\text{max}}$ at every point in time
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Optimum cruise profile (Jet)

Performance diagram - Jet

For basic flight mechanics applications, thrust of a turbojet can be assumed to be constant with airspeed for a given flight altitude.

\[ C_D = C_{D_0} + \frac{C_L^2}{\pi Ae} \]
Optimum cruise profile (Jet)

Performance diagram - Jet

For basic flight mechanics applications, thrust of a turbojet can be assumed to be constant with airspeed for a given flight altitude.

\[ P_a = TV \]
\[ P_r = DV \]
Optimum cruise profile (Jet)

Thrust specific fuel consumption

\[ F = c_T T \]

Additional assumption:
\( c_T \) constant

Specific range

\[ \frac{V}{F} = \frac{V}{c_T T} = \frac{V}{c_T D} \]

\[ \left( \frac{V}{F} \right)_{\text{max}} \quad \text{if} \quad \left( \frac{D}{V} \right)_{\text{min}} \]

What is the corresponding airspeed?
Optimum cruise profile (Jet)

Optimal airspeed for given altitude and weight

Method to calculate the best airspeed

Optimum criterion

Lift over drag ratio

Angle of attack

Airspeed (for given H,W)

\[(D/V)_{\text{min}} \rightarrow C_L^x / C_D^y \rightarrow C_L \rightarrow V\]
Optimum cruise profile (Jet)

Airspeed

1. Optimum criterion

\[
\left( \frac{D}{V} \right)_{\text{min}} \Rightarrow \left( \frac{V}{D} \right)_{\text{max}}
\]

2. Airspeed

\[L = W\]

\[V = \sqrt{\frac{W}{S} \frac{2}{\rho C_L}}\]

3. Drag

\[D = \frac{L}{W} = \frac{C_D}{C_L} W\]

4. Ratio

\[
\frac{V}{D} = \sqrt{\frac{W}{S} \frac{1}{\rho C_L}} = \sqrt{\frac{1}{W \cdot S} \frac{2}{\rho C_D^2}}
\]

5. For a given weight

\[V \propto \sqrt{\frac{C_L}{C_D^2} \frac{1}{\rho}}\]

6. Angle of attack for given altitude

\[
\left( \frac{V}{D} \right)_{\text{max}} \Rightarrow \left( \frac{C_L}{C_D^2} \right)_{\text{max}}
\]
Optimum cruise profile (Jet)

Airspeed

\[
\left( \frac{V}{F} \right)_{\text{max}} \Rightarrow \left( \frac{V}{D} \right)_{\text{max}} \Rightarrow \left( \frac{C_L}{C_D^2} \right)_{\text{max}}
\]

lift–drag–polar

\[ C_{L_{\text{opt}}} = \sqrt{\frac{1}{3} C_{D_0}} \pi Ae \]

Airspeed for given altitude and weight

\[ L = W \]

\[ V = \sqrt{\frac{W}{S} \cdot \frac{2}{\rho} \cdot \frac{1}{C_{L_{\text{opt}}}}} \]

First year:

\[
\left( \frac{C_L}{C_D^2} \right)_{\text{max}} \Rightarrow \frac{d}{dC_L} \left( \frac{C_L}{C_D^2} \right) = 0
\]

\[
C_L \cdot 2C_D \cdot \frac{dC_D}{dC_L} - C_D^2 \cdot 1 = 0
\]

\[
C_D^4 = 0
\]

\[
\frac{dC_D}{dC_L} = \frac{2C_L}{\pi Ae}
\]

\[
C_D^4 \neq 0
\]

\[
2C_L = \frac{1}{2} C_D = \frac{1}{2} \frac{C_{D_0} + C_L^2}{\pi Ae}
\]

\[
C_L = \sqrt{\frac{1}{3} C_{D_0} \pi Ae}
\]
Optimum cruise profile (Jet)

Example question

Question 1 – Cruise flight

The Gulfstream IV, indicated in Figure 1 is a twin-turbofan executive transport aircraft. Data for this aircraft are given below:

\[ S = 88.3 \text{ [m}^3\text{]} \]
\[ b = 23.7 \text{ [m]} \]
\[ A = 6.36 \text{ [-]} \]
\[ e = 0.67 \text{ [-]} \]
\[ C_{D_0} = 0.015 \text{ [-]} \]

\[ C_D = C_{D_0} + \frac{C_L^2}{\pi A e} \]

Figure 1: Gulfstream IV

For jet aircraft, fuel consumption can be represented with the following equation:

\[ F = c_f T \]

Aircraft Weight \( W = 300.000 \text{ [N]} \) (start of cruise)

What is the best airspeed to fly (for max range) at 9000 [m] altitude?\n
\( (\rho = 0.4663 \text{ [kg/m}^3\text{]}, \ T = 229.65 \text{ [K]} \)
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Effect of altitude

Performance diagram

Increasing $H$

Angle of attack is constant for a given point on the drag curve

\[ V = \sqrt{\frac{W}{S}} \frac{2}{\rho C_L} \propto \frac{1}{\sqrt{\rho}} \]

\[ D = \frac{C_D}{C_L} W \propto \rho^0 \]
Effect of altitude

Specific range

Altitude 0 m

Optimum cruise level

$T_{\text{at } \Gamma_{\text{cruise}}}$
Effect of altitude

Conclusion

At increasing altitude:
- V/F increases
- V increases
- Engine more efficient

Thus: fly as high as possible! (up to the limits of the engine)
Effect of altitude

- Optimum cruise level at $\Gamma_{\text{cruise}}$
- Theoretical ceiling at $\Gamma_{\text{cruise}}$

$H_{\text{cr}} < H_{\text{th}}$ (at $\Gamma_{\text{cruise}}$)

Optimum $H_{\text{cr}} \approx H_{\text{s}}$ (service ceiling)
Effect of altitude

In the presence of speed limits (e.g. $M_{MO}$)

Increasing $H$

$V_{lim}$

$D$

$V$
Effect of altitude
In the presence of speed limits (e.g. $M_{MO}$)

- Optimum V/F at $V = V_{lim}$

$$D_{min} \Rightarrow \left( \frac{C_L}{C_D} \right)_{max} \Rightarrow C_L = \sqrt{C_{D0} \pi Ae}$$
Summary – Jet aircraft

- Choose $V$ such that $(V/F)_{\text{max}} \rightarrow (C_L / C_D^2)_{\text{max}}$

- $H$ as high as possible (limited by the engine)

- If the speed limit is reached at lower altitude:
  - $V = V_{\text{lim}}$
  - $H$ is such that $C_L / C_D$ is max
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Effect of weight

\[
V = \sqrt{\frac{W}{S}} \frac{2}{\rho} \frac{1}{C_L} \propto \sqrt{W}
\]

\[
D = \frac{C_D}{C_L} W \propto W
\]

\[
P_r = DV \propto W \sqrt{W}
\]

\[\Rightarrow\]

\[D \propto V^2 \text{ at constant } \alpha\]

\[P_r \propto V^3 \text{ at constant } \alpha\]
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Best flying strategy

**Strategies:**
- I: constant altitude and engine setting
- II: constant altitude and airspeed
- III: constant angle of attack

\[
W_1, H_1 \quad W_2 < W_1, H_1 \quad W_2, H_2 > H_1
\]
Best flying strategy

**Constant altitude**

I. $\Gamma = \text{constant}: \ (V/F) \ll (V/F)_{\text{opt}} \text{ but } V \uparrow$

II. $V = \text{constant} \ (V/F) < (V/F)_{\text{opt}}$

III. $\alpha = \text{constant}, \ V \downarrow \text{ but } (V/F) = (V/F)_{\text{opt}}$

**Climb**

IV. $\alpha = \text{constant}, \ V = \text{constant}, \ (V/F) = (V/F)_{\text{opt}}, \text{ even } > (V/F)_0$

V. $\alpha = \text{constant}, \text{ changing } V?$
Best flying strategy

Optimum cruise climb possible?

**Strategy IV** → $\alpha_2 = \alpha_1 \ (C_L \text{ is constant})$ and $V_2 = V_1$

$$V_1 = \sqrt{\frac{W_1 \ 2 \ 1}{S \ \rho_1 \ C_L}}$$

Is this possible? Are the engines capable of providing enough thrust at higher altitude and lower weight?
Best flying strategy

Typical turbojet performance

\[ \frac{T}{T_0} = \left( \frac{\rho}{\rho_0} \right)^{0.75} \]

\[ \frac{T}{T_S} = \frac{\rho}{\rho_S} \]
Best flying strategy

Optimum cruise climb possible?

**Strategy IV** → $\alpha_2 = \alpha_1$ ($C_L$ is constant) and $V_2 = V_1$

\[
\frac{W}{\rho} = \text{constant}
\]

- This is exactly how a typical turbojet behaves above 11km. So there will be enough thrust.
- Strategy V is not feasible
- Below 11km there will be enough thrust as well
Best flying strategy

Optimum cruise climb possible?

**Strategy IV** $\Rightarrow \alpha_2 = \alpha_1$ ($C_L$ is constant) and $V_2 = V_1$

The engines can provide just enough thrust
(strategy V not possible)

What happens in case of $M_{lim}$?

Mach number is constant at constant airspeed above 11km
$\Rightarrow$ No problem
Best flying strategy

- **takeoff weight**
- **climb**
- **cruise**
- **descent**
- **landing weight**

**range R**

**total range (stage length)**

**altitude**
Best flying strategy
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Analytic range equations

Breguet range equation

- Range

\[
\frac{dW}{dt} = \frac{dW}{ds} V = -F
\]

\[
R = \int_{s_0}^{s_1} ds = \int_{w_0}^{w_1} \frac{V}{F} dW
\]
Analytic range equations

Breguet range equation for jet aircraft

• Jet aircraft optimum climb cruise ($\alpha$, $V$ and $c_T$ are constant during variation of $W$

\[
R = \int_{w_i}^{w_0} \frac{V}{F} dW
\]

\[
R = \int_{w_i}^{w_0} \frac{V}{c_T D} dW
\]

\[
R = \int_{w_i}^{w_0} \frac{V}{c_T C_D} \frac{C_L}{W} dW
\]

\[
R = \frac{V}{c_T C_D} \int_{w_i}^{w_0} \frac{W_0}{W} dW
\]

\[
R = \frac{V}{c_T C_D} \ln \left( \frac{W_0}{W_1} \right)
\]
Analytic range equations

Breguet range equation for jet aircraft

\[ R = \frac{V}{c_T} \frac{C_L}{C_D} \ln \left( \frac{W_0}{W_1} \right) \]

- If \( V \) is not limited: \( R_{\text{max}} \) at \( (V \frac{C_L}{C_D})_{\text{max}} \rightarrow (\frac{C_L}{C_D^2})_{\text{max}} \) and \( \rho_{\text{min}} \)

- If \( V \) is limited:
  \( R_{\text{max}} \) at \( V = V_{\text{lim}} \) and \( \rho \) such that \( (\frac{C_L}{C_D})_{\text{max}} \)
Analytic range equations

Breguet range equation for propeller aircraft

\[ F = c_p P_{br} \]

\[ \frac{V}{F} = \frac{\eta_j}{c_p T} \]

Cruise flight with constant \( \alpha \), \( c_p \) and \( \eta_j \):

\[ R = \int_{w_1}^{w_0} \frac{V}{F} dW \]

\[ R = \frac{\eta_j C_L}{c_p C_D} \ln \left( \frac{W_0}{W_1} \right) \]
Analytic range equations

Breguet range equation for propeller aircraft

Conclusion: Altitude is not important w.r.t V/F
But V is larger at high altitude
Analytic range equations

Unified Breguet range equation

- Jet aircraft
  \[ \eta_{tot} = \frac{TV}{H \frac{F}{g}} \]

- Propeller aircraft
  \[ \eta_{tot} = \frac{TV}{H \frac{F}{g}} \]
# Analytic equations

## Unified Breguet range equation

<table>
<thead>
<tr>
<th></th>
<th>Time</th>
<th>1920</th>
<th>Lindbergh</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>43000 kJ/kg</td>
<td>43000 kJ/kg</td>
<td>43000 kJ/kg</td>
<td></td>
</tr>
<tr>
<td>( \eta_{\text{tot}} )</td>
<td>0.20</td>
<td>0.20 – 0.30</td>
<td>&gt;0.40</td>
<td></td>
</tr>
<tr>
<td>L/D</td>
<td>10</td>
<td>11</td>
<td>16-18</td>
<td></td>
</tr>
<tr>
<td>( W_1/W_0 )</td>
<td>0.6 – 0.7</td>
<td>0.5</td>
<td>0.5</td>
<td></td>
</tr>
</tbody>
</table>
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Story

History

- 1919: Alcock / Brown: Newfoundland → Ireland
- Fonck, Nungesser/Coli, Lindbergh: New York → Paris

<table>
<thead>
<tr>
<th>Overload</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Fuel</td>
<td></td>
</tr>
<tr>
<td>Passengers</td>
<td></td>
</tr>
<tr>
<td>OEW (Operational Empty Weight)</td>
<td></td>
</tr>
</tbody>
</table>

- Structural safety factor
- Take–off length ($W^2$)
- Climb gradient after take off
- Tailwind west → east
Story

Spirit of St. Louis

- Charles Lindbergh, 1927
- First solo, nonstop flight across the Atlantic Ocean
Story

Spirit of St. Louis

- Charles Lindbergh had to decrease airspeed to achieve maximum range

![Diagram showing the relationship between $P_r$ and $V$ for decreasing weight](image)
N.A.C.A. Technical Note No. 257

Fig. 9

237 B.H.P. available at 1850 R.P.M. (flight test)

B.H.P. required, 5130 lb. gross wt., Full load, 425 gal. gasoline, 25 gal. oil.

B.H.P. required, 3770 lb. gross wt., Half gas used.

B.H.P. required, 2415 lb. gross wt., Light load, No gasoline, 10 gal. oil.

Air speed, M.P.H.

Fig. 9 Ryan NYP airplane.
Story

Global flyer
Story

Global flyer

- First part of flight: insufficient strength to withstand gusts
- Best glide ratio: 1:37
  - \( \frac{C_L}{C_D} \)_{max} = 37
  - \( C_{D0} = 0.018 \) \( e = 0.85 \)
- \( H_{cr} = 45000 \text{ ft} = 13.716 \text{ m} \) \( \rightarrow \rho = 0.2377 \)
- Distance flown 38000 km
- Time 66 hrs
- Fuel lost 2600 lbs \( \rightarrow \) actual fuel fraction 71%
Story

Global flyer

- \((V/F)_{\text{max}}\): \(C_L = \sqrt{\frac{1}{3} C_D \pi Ae} = 0.72\)
  \(C_D = 0.024 \Rightarrow \frac{C_L}{C_D} = 30\)

- \(V\) for \((V/F)_{\text{max}}\), 45000 ft, \(W_{\text{gross}}\): \(V = 175\) m/s
- Time for 40,000 km at constant \(V\): 64 hrs
- Guesstimate of \(\eta_{\text{tot}}\):
  - High bypass fans at 1000 km/h: \(\eta_{\text{tot}} = 40\%\)
  - Medium bypass fans \(\eta_{\text{tot}} = 35\%, \eta_{\text{th}} = 50\%, \eta_j = 70\% \Rightarrow V_j / V = 1.86\)
- Correction for lower flight speed:
  - \(V_j / V = 3 \Rightarrow \eta_j = 0.5 \Rightarrow \eta_{\text{tot}} = 25\%\)

- Range in ideal climbing cruise: \(R = 53000\) km
Story

Global flyer

• Cruise at $V = \text{constant}$ and $H_{cr} = \text{constant}$: $R = 37.500 \text{ km}$

• At fuel fraction 70%: $R = 28000 \text{ km}$

• Influence wind?
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Weight breakdown

- **Wide-body airplane with turbofan engines**
  - Total fuel
  - Payload
  - OEW

- **Supersonic transport with turbojet engines**
  - Total fuel
  - Payload
  - OEW

*Useful load*
Weight breakdown

**Payload range diagram**

MTOW = maximum take-off weight
MZFW = maximum zero fuel weight
OEW = operational empty weight
Weight breakdown

Maximum zero fuel weight

MZFW limited, amongst others by bending moment of the wing

MTOW > MZFW at same bending moment. MTOW limited e.g. by landing gear
Weight breakdown

Reserve fuel

• Reserve fuel
  • In general:
    • Fuel to alternate
    • 45 minutes holding at altitude

• Fuel shortage:
  • In general:
    • Management problem
    • CRM Cockpit resource management
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Economics

Block time and block speed

Key Parameters

Range $R$
Payload $P$
Block time $E_B$
Block speed $V_B$
Transport product $P_R$
Transport productivity $P_h$
Revenue earning capacity $P_y$
Economics
Economics

Transport productivity

Conclusion:
Maximum transport productivity is achieved at the design range

![Cost (direct operating cost) must be considered as well of course]
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Summary

The **objective** is to minimize fuel for a given range

Key parameter:
Specific range $V/F$

$[V]/[F] = [m/s]/[kg/s] = [m/kg]$  
So it is the distance travelled per unit of fuel

*Performance diagram for jet aircraft*
Summary

Effect of weight and altitude
Summary

Key conclusions

Jet aircraft (analytical approximation)
1. Choose $V$ such that $(V/F)_{\text{max}} \rightarrow (C_L / C_D^2)_{\text{max}}$
2. $H$ as high as possible (limited by the engine)
3. If the speed limit is reached at lower altitude: $V = V_{\text{lim}}$, $H$ is such that $C_L / C_D$ is max

Propeller aircraft (analytical approximation)
1. Conclusion: Altitude is not important w.r.t $V/F$
2. But $V$ is larger at high altitude
Summary

Breguet range equation

\[ R = \int_{W_0}^{W_1} ds = \int_{W_i}^{W_0} \frac{V}{F} dW \]

Jet aircraft
(analytical approximation)

\[ R = \int_{W_i}^{W_0} \frac{V}{C_L} \frac{C_T}{C_D} \frac{dW}{W} \]

Propeller aircraft
(analytical approximation)

\[ R = \int_{W_i}^{W_0} \frac{\eta_j}{C_L} \frac{C_T}{C_D} \frac{dW}{W} \]

Optimum cruise climb

\[ R = \frac{V}{C_L} \frac{C_T}{C_D} \ln \left( \frac{W_0}{W_1} \right) \]

Cruise flight with constant \( \alpha, c_p \) and \( \eta_j \):

\[ R = \frac{\eta_j}{C_L} \frac{C_T}{C_D} \ln \left( \frac{W_0}{W_1} \right) \]
Summary

Unified Breguet range equation

- Jet aircraft
  \[ \eta_{tot} = \frac{TV}{H} \frac{F}{g} = \frac{V}{c_T} \frac{g}{H} \]

- Propeller aircraft
  \[ \eta_{tot} = \frac{TV}{H} \frac{F}{g} = \frac{\eta_j}{c_p} \frac{g}{H} \]

Fuel quality  Propulsion efficiency  aerodynamic quality

- Both:
  \[ R = \frac{H}{g} \eta_{tot} \frac{C_L}{C_D} \ln \left( \frac{W_0}{W_1} \right) \]

Structural characteristics
Questions?