MATH 31A (Butler)

Practice for Midterm IIa (Solutions)

1. (a) Verify that $(2,1)$ is a critical point for the curve $y^3x - 3yx^2 = x^2 - 15x + 16$. (You need to verify two things: (i) it is on the curve and (ii) it is a critical point.)

> To verify it is a point on the curve we plug $x = 2$ and $y = 1$ into both sides and we get
> $$1^3{\cdot}2 - 3{\cdot}1{\cdot}2^2 = -10 = 2^2 - 15{\cdot}2 + 16.$$
> So it is a point on the curve. To see that it is a critical point we first compute the derivative using implicit differentiation, i.e.,
> $$3y^2y'x + y^3 - 3y'x^2 - 6xy = 2x - 15 \quad \text{or} \quad (3y^2x - 3x^2)y' = 2x + 6xy - 15 - y^3.$$
> Plugging $x = 2$ and $y = 1$ into this equation we have
> $$(3{\cdot}1^2{\cdot}2 - 3{\cdot}2^2)y' = 2{\cdot}2 + 6{\cdot}2{\cdot}1 - 15 - 1^3 \quad \text{or} \quad -6y' = 0,$$
> and so we can conclude at $x = 2$ and $y = 1$ that $y' = 0$, i.e., it is a critical point.

(b) Use the second derivative test to determine if the point $(2,1)$ is a maximum or a minimum.

> To use the second derivative test we need the second derivative. So starting with
> $$(3y^2x - 3x^2)y' = 2x + 6xy - 15 - y^3$$
> (from above) we take the derivative of both sides to get the following
> $$(6yy'x + 3y^2 - 6x)y' + (3y^2x - 3x^2)y'' = 2 + 6xy' + 6y - 3y^2y'.$$
> Now we can substitute in $x = 2$, $y = 1$ and $y' = 0$ so that we have
> $$(6{\cdot}1{\cdot}0{\cdot}2 + 3{\cdot}1^2 - 6{\cdot}2){\cdot}0 + (3{\cdot}1^2{\cdot}2 - 3{\cdot}2^2)y'' = 2 + 6{\cdot}2{\cdot}0 + 6{\cdot}1 - 3{\cdot}1^2{\cdot}0,$$
> which simplifies to $-6y'' = 8$ so that $y'' = -4/3 < 0$. So by the second derivative test the curve is concave down at $(2,1)$ and so $(2,1)$ is a maximum.

2. You have recently started a new job as quality control officer in the new widget factory, company slogan: "You fidget, we widget". A new machine has been installed that takes a rod of length x inches for x between 1 and 10 and returns a widget of size $x^2 + 3x + 2$. Ideally, you want all of your widgets to have an output of size 20, which would correspond to starting with a rod of exactly 3 inches, but you know that the rods are not exactly 3 inches.

Given that you want the widgets to be within $\frac{1}{2}$ of 20 (i.e., you can have widgets of sizes in the range of $20 \pm \frac{1}{2}$), estimate how much error you can have for the length of the rods that go in.

We are trying to get an estimate at how much our input can vary so that our output is within a certain range. Or in other words we are trying to get an estimate for how much we can vary x by (Δx) given that we want to control how much we want to control how $f(x)$ varies $(\Delta f = \pm\frac{1}{2})$. We have that

$$\Delta f \approx f'(a)\Delta x$$

Since $f'(x) = 2x + 3$ and we are interested in $x = 3$ (i.e., so $a = 3$) then we have

$$\pm\frac{1}{2} \approx f'(3)\Delta x = 9\Delta x \quad \text{or} \quad \Delta x \approx \pm\frac{1}{18}.$$

In other words if we want to have the output be within $\frac{1}{2}$ of 20 then we should have the input be within $\pm\frac{1}{18}$ inches of 3 inches.

3. (a) After several months as quality control officer you have been promoted and are now an executive. After changing the company slogan to "Don't be a nidget! Get the best widget!", you have been asked to decide how many widgets to produce in order to maximize the profit (which is the difference between revenue and cost).

If w is the number of widgets then the cost of producing the widgets is $90 + \frac{1}{10}w^2$ dollars and each widget can be sold for for \$8, how many widgets should you produce, and how much profit will the company make?

The profit is revenue ($8w$, i.e., we get 8 dollars for every widget we make and sell) minus the cost ($90 + \frac{1}{10}w^2$). So that as a function of w we have

$$P(w) = 8w - (90 + \frac{1}{10}w^2) \text{ and so } P'(w) = 8 - \frac{1}{5}w.$$

Since the derivative is always defined then setting this to 0 and solving we have that the critical point is $w = 40$. Since $P''(w) = -\frac{1}{5}$ then by the second derivative test this point is a maximum and so will indeed give the maximum profit.

Finally, to find the amount of profit we make we plug it back into the original function giving

$$P(40) = 8{\cdot}40 - (90 + \frac{1}{10}40^2) = 320 - (90 + 160) = 70 \text{ dollars.}$$

(b) At what price for widget will the company only be able to break even at best?

In other words the maximum profit will be 0. Let q be the price of a widget, then similar to part (a) we have that the profit is

$$P(w) = qw - (90 + \frac{1}{10}w^2) \text{ and so } P'(w) = q - \frac{1}{5}w.$$

In particular, for any price point q we will produce $w = 5q$ widgets. Our profit at this production level will be

$$P(5q) = 5q^2 - (90 + \frac{1}{10}25q^2) = \frac{5}{2}q^2 - 90.$$

We will break even at best then if $P(5q) = 0$ which corresponds to

$$\frac{5}{2}q^2 = 90 \text{ or } q^2 = 36 \text{ or } q = 6.$$

So at a price of 6 dollars we break even. (Below 6 dollars the widget company would go out of business!)

4. (a) On what intervals is $h(x) = x^2 - 2\arctan(x^2)$ increasing and on what intervals is it decreasing?

To tell if a function is increasing or decreasing we look at the first derivative. So we have using the chain rule and the derivative of the arctangent function

$$h'(x) = 2x - 2\frac{1}{1+(x^2)^2}\cdot 2x = 2x\left(1 - \frac{2}{1+x^4}\right) = 2x\left(\frac{x^4-1}{1+x^4}\right).$$

The denominator is never 0 and so the derivative is never undefined. Looking at the numerator we see that it will be equal to 0 when $x = 0$ or $x^4 = 1$, i.e., $x = \pm 1$. So we have three critical points $x = -1, x = 0, x = 1$. We now test points from the various intervals to determine whether the derivative is positive or negative (indicating if the function is increasing or decreasing).

$$h'(-2) = \frac{(-4)\cdot(15)}{17} < 0 \quad \longleftarrow \quad \text{decreasing for } x \le -1$$

$$h'(-\frac{1}{2}) = \frac{(-1)\cdot(-\frac{15}{16})}{\frac{17}{16}} > 0 \quad \longleftarrow \quad \text{increasing for } -1 \le x \le 0$$

$$h'(\frac{1}{2}) = \frac{(1)\cdot(-\frac{15}{16})}{\frac{17}{16}} < 0 \quad \longleftarrow \quad \text{decreasing for } 0 \le x \le 1$$

$$h'(2) = \frac{(4)\cdot(15)}{17} > 0 \quad \longleftarrow \quad \text{increasing for } 1 \le x$$

(b) Find the global maximum and global minimum for $h(x) = x^2 - 2\arctan(x^2)$ on the interval $0 \le x \le \sqrt[4]{3}$. (Hint: $h(\sqrt[4]{3}) = \sqrt{3} - \frac{2}{3}\pi \approx -0.362344$.)

To find the global maximum on a closed interval we make a list of the boundary points (0 and $\sqrt[4]{3}$) and the critical points in the interval (0 and 1 using part (a)). We now test at each one of these three points.

$$\begin{aligned} h(0) &= 0^2 - 2\arctan(0^2) = 0 \\ h(1) &= 1^2 - 2\arctan(1^2) = 1 - \frac{1}{2}\pi \quad \longleftarrow \quad \text{this is } < 1 - \tfrac{1}{2}3 = -\tfrac{1}{2} \\ h(\sqrt[4]{3}) &= \sqrt{3} - \frac{2}{3}\pi \approx -0.362344 \quad \longleftarrow \quad \text{by the hint} \end{aligned}$$

So we see that the maximum value is 0 and occurs at $x = 0$ and the minimum value is $1 - \frac{1}{2}\pi$ and occurs at $x = 1$.

5. Consider the functions $f(x) = 2\tan x + 2\sec x$ and $g(x) = (\tan x + \sec x)^2 + 1$. Is $g(x)$ the derivative of $f(x)$ or is $g(x)$ the anti-derivative of $f(x)$? Justify your answer. (Hint: $\sec^2 x = \tan^2 x + 1$.)

We have that

$$\begin{aligned} f'(x) &= 2\sec^2 x + 2\sec x \tan x \\ &= \sec^2 x + 2\sec x \tan x + \sec^2 x \\ &= \tan^2 x + 2\sec x \tan x + \sec^2 x + 1 \\ &= (\tan x + \sec x)^2 + 1 \;=\; g(x), \end{aligned}$$

so that $g(x)$ is the derivative of $f(x)$.
It is also possible that $g(x)$ could be the anti-derivative of $f(x)$ but in this case it is not true. To see this we note that

$$g'(x) = 2(\tan x + \sec x)\big(\sec^2 x + \sec x \tan x).$$

But this is not $f(x)$ since (for example)

$$g'(\frac{\pi}{4}) = 2(1+\sqrt{2})(2+\sqrt{2}) \neq 2 + 2\sqrt{2} = f(\frac{\pi}{4}).$$