MATH 31A (Butler)

Practice for Midterm I (Solutions)

1. (a) Find $\lim_{t\to 0} \dfrac{\left(\dfrac{1}{\sqrt{9+2t}} - \dfrac{1}{3}\right)}{\sin(3t)}$.

Let us start by simplifying what is inside the limit. This will be done by adding fractions, and then multiplying by the conjugate.

$$\begin{aligned}
\frac{\left(\dfrac{1}{\sqrt{9+2t}} - \dfrac{1}{3}\right)}{\sin(3t)} &= \frac{\left(\dfrac{3}{3\sqrt{9+2t}} - \dfrac{\sqrt{9+2t}}{3\sqrt{9+2t}}\right)}{\sin(3t)} \\
&= \frac{3-\sqrt{9+2t}}{3\sqrt{9+2t}\sin(3t)} \\
&= \frac{3-\sqrt{9+2t}}{3\sqrt{9+2t}\sin(3t)} \cdot \frac{3+\sqrt{9+2t}}{3+\sqrt{9+2t}} \\
&= \frac{9-(9+2t)}{3\sqrt{9+2t}\sin(3t)(3+\sqrt{9+2t})} \\
&= \frac{-2t}{3\sqrt{9+2t}\sin(3t)(3+\sqrt{9+2t})} \\
&= \frac{-2}{9\sqrt{9+2t}\left(\dfrac{\sin(3t)}{3t}\right)(3+\sqrt{9+2t})}.
\end{aligned}$$

So applying basic rules of limits we have

$$\begin{aligned}
\lim_{t\to 0}\frac{\left(\dfrac{1}{\sqrt{9+2t}} - \dfrac{1}{3}\right)}{\sin(3t)} &= \lim_{t\to 0}\frac{-2}{9\sqrt{9+2t}\left(\dfrac{\sin(3t)}{3t}\right)(3+\sqrt{9+2t})} \\
&= \frac{-2}{9\lim_{t\to 0}\sqrt{9+2t}\left(\displaystyle\lim_{t\to 0}\dfrac{\sin(3t)}{3t}\right)\displaystyle\lim_{t\to 0}(3+\sqrt{9+2t})} \\
&= \frac{-2}{9\cdot 3\cdot 1\cdot 6} \\
&= -\frac{1}{81}.
\end{aligned}$$

(b) Find $\lim_{x\to 0}\left(\frac{1}{3x}-\frac{1}{x(x+3)}\right)$.

Again we start by simplifying what is inside the limit. we have

$$\frac{1}{3x}-\frac{1}{x(x+3)}=\frac{(x+3)}{3x(x+3)}-\frac{3}{3x(x+3)}=\frac{x}{3x(x+3)}=\frac{1}{3(x+3)}.$$

And so we have

$$\lim_{x\to 0}\left(\frac{1}{3x}-\frac{1}{x(x+3)}\right)=\lim_{x\to 0}\frac{1}{3(x+3)}=\frac{1}{3\cdot 3}=\frac{1}{9}.$$

(Note in the last step we can just plug the point that we are interested in since $1/\big(3(x+3)\big)$ is a continuous function.)

2. Let $g(x)=\begin{cases} x\cos x-2x+3 & \text{if } x\le 0;\\ bx+3 & \text{if } x>0.\end{cases}$

(a) Show that $g(x)$ is continuous at $x=0$ for *any* value of b.

The function will be continuous at 0 if

$$\lim_{x\to 0} g(x)=g(0).$$

From the definition it is easy to see that $g(0)=3$. So we need to check the limit. Since this is a piecewise function it is best to split the limit into two parts (from the left and right). If they both are 3 then the function is continuous. And so we have

$$\begin{aligned}\lim_{x\to 0-} g(x) &= \lim_{x\to 0-}\big(x\cos x-2x+3\big)=3\\ \lim_{x\to 0+} g(x) &= \lim_{x\to 0+}\big(bx+3\big)=3.\end{aligned}$$

(By splitting into the left and right limits we can substitute in the appropriate definition of $g(x)$ in the region, since each one of these are continuous functions we can now simply plug in the point we are taking the limit to, in this case 0 to find the value of the limit.) In particular, since the limit from the right was always 3 regardless of the value of b we can conclude that the limit as $x\to 0$ of $g(x)$ is 3 and so the function is continuous at 0.

(b) For what value of b does $g'(0)$ exist?

Let us attempt to calculate what $g'(0)$ should be. We can do this by using the definition of the derivative. This would be

$$g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{g(h) - 3}{h}.$$

Again it is best to break this into cases, i.e., from below or above. So we have

$$\lim_{h\to 0+} \frac{g(h) - 3}{h} = \lim_{h\to 0+} \frac{\big(bh + 3\big) - 3}{h} = \lim_{h\to 0+} b = b.$$

While

$$\lim_{h\to 0-} \frac{g(h) - 3}{h} = \lim_{h\to 0-} \frac{\big(h\cos h - 2h + 3\big) - 3}{h} = \lim_{h\to 0-} \big(\cos h - 2\big) = 1 - 2 = -1.$$

We need to have the limit from above equal the limit from below. So we can conclude we need $b = -1$.

(c) For the answer in part (b), what is the value of $g'(0)$?

From looking at the limit from part (b) for calculating $g'(0)$ we see that

$$g'(0) = b = -1.$$

3. Find the tangent line of $y = 2x + 4\sqrt{x} - \pi^2$ parallel to the line $y = 3x + 7$.

We start by recalling that two lines are parallel if they have the same *slope*. So we need to find all tangent lines with a slope of 3, and this can be done by setting $y' = 3$ and solving for x. We have

$$y = 2x + 4x^{1/2} - \pi^2 \text{ and so we get } y' = 2 + 2x^{-1/2} = 3.$$

(Note the derivative of π^2 (a constant) is 0 and not 2π.) Rearranging, this becomes

$$\frac{2}{\sqrt{x}} = 1 \text{ or } \sqrt{x} = 2 \text{ or } x = 4$$

So the graph has a slope of 3 when $x = 4$ which corresponds to

$$y = 2\cdot 4 + 4\sqrt{4} - \pi^2 = 16 - \pi^2.$$

Now with out point and slope it is easy to find the line using the equation

$$y = f(a) + f'(a)(x - a) = \big(16 - \pi^2\big) + 3(x - 4) = 3x + 4 - \pi^2.$$

4. Express $\dfrac{d^2}{dx^2}\big(f(x)g(x)\big)$ in terms of $f(x)$, $f'(x)$, $f''(x)$, $g(x)$, $g'(x)$ and $g''(x)$.

This problem tests our ability to use the product rule. We have

$$\begin{aligned}
\frac{d^2}{dx^2}\big(f(x)g(x)\big) &= \frac{d}{dx}\left(\frac{d}{dx}\big(f(x)g(x)\big)\right) \\
&= \frac{d}{dx}\big(f'(x)g(x) + f(x)g'(x)\big) \\
&= \frac{d}{dx}\big(f'(x)g(x)\big) + \frac{d}{dx}\big(f(x)g'(x)\big) \\
&= \big(f''(x)g(x) + f'(x)g'(x)\big) + \big(f'(x)g'(x) + f(x)g''(x)\big) \\
&= f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x).
\end{aligned}$$

5. Given that $f(1) = 2$, $f'(1) = -1$, $g(1) = 3$ and $g'(1) = 2$ find $h(1)$ and $h'(1)$ where $h(x) = x^2 f(x) - 3\sqrt{x}g(x)$.

First let us find an expression for $h'(x)$. Again this is applying the basic rules of derivatives; namely the linearity of derivatives and the product tule. So we have

$$\begin{aligned}
h'(x) = \frac{d}{dx}\big(x^2 f(x) - 3\sqrt{x}g(x)\big) &= \frac{d}{dx}\big(x^2 f(x)\big) - 3\frac{d}{dx}\big(x^{1/2}g(x)\big) \\
&= \big(2xf(x) + x^2 f'(x)\big) - 3\big(\frac{1}{2}x^{-1/2}g(x) + x^{1/2}g'(x)\big).
\end{aligned}$$

And so we have

$$\begin{aligned}
h(1) &= 1^2 f(1) - 3\sqrt{1}g(1) = 2 - 3{\cdot}3 = -7 \\
h'(1) &= \big(2f(1) + f'(1)\big) - 3\big(\frac{1}{2}g(1) + g'(1)\big) = \big(2{\cdot}2 + (-1)\big) - 3\big(\frac{1}{2}3 + 2\big) \\
&= 3 - \frac{21}{2} = -\frac{15}{2}
\end{aligned}$$

6. The position of a particle on a strip is given by $s(t) = t^3 - 5t^2 + 6t - 4$ ($s(t)$ is measured in inches and t is measured in seconds).

(a) What is the position of the particle at $t = 1$? (Give the units.)

This one is easy. We find it by computing

$$s(1) = 1^3 - 5{\cdot}1^2 + 6{\cdot}1 - 4 = -2 \text{ inches.}$$

(b) What is the (instantaneous) velocity of the particle at $t = 1$? (Give the units.)

Given position to find velocity we take the derivative. Note the units of the velocity will be the units of the position (in our case inches) divided by the units of time (in our case seconds). So we first compute the derivative

$$s'(t) = 3t^2 - 10t + 6.$$

And so the velocity at $t = 1$ is

$$s'(1) = 3{\cdot}1^2 - 10{\cdot}1 + 6 = -1 \, \frac{\text{inches}}{\text{second}}.$$

(Note the negative value indicates direction, in this case we would say that the particle is moving to the left along the strip.)

(c) What is the acceleration of the particle at $t = 1$? (Give the units.) Does this mean the speed the particle is moving is going up or down?

Given velocity to find acceleration we take the derivative (so that acceleration is the second derivative of position). Note the units of the acceleration will be the units of the velocity (in our case inches per second) divided by the units of time (in our case seconds). So we first compute the derivative

$$s''(t) = 6t - 10.$$

And so the acceleration at $t = 1$ is

$$s''(1) = 6{\cdot}1 - 10 = -4 \, \frac{\text{inches}}{(\text{second})^2}.$$

Now the second part of this question asks about speed. Physicists measure speed in terms of the absolute value (or magnitude) of velocity. Our velocity is negative and the the acceleration is negative which means that around 1 the velocity will become even more negative. So in terms of the speed the speed will be going up at $t = 1$ (note that it was not enough just to look at $s''(1)$ to get our answer, it also depended on $s'(1)$).