MATH 31A (Butler)
Practice for Final (C)

Try to answer the following questions without the use of book, notes or calculator; but you can use the equation sheet posted on the course website. Time yourself and try to finish the questions in less than three hours.

1. Find the area of the region bounded between the curves $g(t) = \sin(\pi t)$ and $f(t) = 8t^2 - 2t$. (Hint: the best way to find the intersection points is to "guess" some easy values for t.)

Since we are not given bounds we first need to figure out the bounds. This reduces to solving for t in the following equation.

$$\sin(\pi t) = 8t^2 - 2t$$

There is *no* method to solve this analytically, and since no calculators are allowed then it must be that we can guess solutions. For example $t = 0$ gives zero on both sides and so that is a point of intersection. The next "nice" value for $\sin(\pi t)$ would occur at $t = 1/2$ and if we plug that in we see that we get 1 on both sides. It is not hard to see that these are the only two points of intersection (if in doubt do a rough sketch of these two functions). Now let us try to see which function is on top, if we pick $t = 1/4$ (a point in the middle) we see that $g(1/4) = \sin(\pi/4) = \sqrt{2}/2$ while $f(1/4) = 8(1/4)^2 - 2(1/4) = 0$, so $g(t)$ is the function on the top and $f(t)$ is the function on the bottom (this last part is easy to see if you do a rough sketch of the functions).
So we have

$$\begin{aligned}
\text{Area} &= \int_0^{1/2} \big(g(t) - f(t)\big)\,dt \\
&= \int_0^{1/2} \big(\sin(\pi t) - (8t^2 - 2t)\big)\,dt \\
&= \int_0^{1/2} \big(\sin(\pi t) - 8t^2 + 2t\big)\,dt \\
&= \left(-\frac{1}{\pi}\cos(\pi t) - \frac{8}{3}t^3 + t^2\right)\Bigg|_{t=0}^{t=1/2} \\
&= \left(-\frac{1}{\pi}\cos\left(\frac{\pi}{2}\right) - \frac{8}{3}\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2\right) - \left(-\frac{1}{\pi}\cos(0) - 0 + 0\right) \\
&= \frac{1}{\pi} - \frac{1}{12}.
\end{aligned}$$

2. (a) For $a \geq \frac{1}{2}$ find the point on the curve $y = \sqrt{x}$ closest to the point $(a, 0)$.

First we note that the function $y = \sqrt{x}$ is only defined for $x \geq 0$ and so throughout we will assume that $x \geq 0$. Let $(x, y) = (x, \sqrt{x})$ be a point on the curve, then the distance between this point and $(a, 0)$ is

$$\sqrt{(x-a)^2 + (\sqrt{x}-0)^2} = \sqrt{x^2 - 2ax + a^2 + x} = \sqrt{x^2 - (2a-1)x + a^2}.$$

Since we are trying to minimize this expression, it is the same as trying to minimize the expression inside the square root, i.e., minimize $g(x) = x^2 - (2a-1)x + a^2$. We have

$$g'(x) = 2x - (2a-1)$$

which is never undefined and is equal to 0 at

$$2x = 2a - 1 \quad \text{or} \quad x = a - \frac{1}{2}.$$

(Note that $a - \frac{1}{2} \geq 0$ and so this is still a point on the curve.) By taking the second derivative we have $g''(a - 1/2) = 2 > 0$ showing that this is indeed a minimum. Therefore we can conclude that the point on the curve closest to $(a, 0)$ is at the point

$$\left(a - \frac{1}{2}, \sqrt{a - \frac{1}{2}}\right).$$

(b) What is the closest point on the curve $y = \sqrt{x}$ for the point $(a, 0)$ when $a < \frac{1}{2}$?

Looking at our analysis for part (a) we see that when $a < \frac{1}{2}$ that

$$g'(x) = 2x + 2\underbrace{\left(\frac{1}{2} - a\right)}_{>0} > 0$$

for all $x \geq 0$. In particular, we see that the minimum for $g(x)$ occurs at the endpoint $x = 0$. So in the case that $a < \frac{1}{2}$ then the closest point on the curve $y = \sqrt{x}$ to the point $(a, 0)$ is $(0, 0)$.

3. To help practice your timing on the final you have started using a large hourglass when doing practice problems. But before long you discover that it is much more interesting to watch the hourglass than to work on the practice problems! In particular, you notice that the sand in the top half of the hourglass forms a cone shape where sand drops out of the tip of the cone into the bottom half of the hourglass. By repeated playing around with the sand you know that the sand drops out of the top half of the hourglass at a constant rate of 2π cubic inches per minute. You also see that when you first flip the hourglass over that the cone formed by the sand has a height of five inches and is four inches across at the top.

At what height will the depth of the sand be when the sand is dropping at a rate of two inches per minute? (Hint: the volume of a cone is $V = \frac{1}{3}\pi r^2 h$ where r is the radius and h is the height.)

First we note that we can take the volume and rewrite it completely in terms of h. This is because as the sand drops out of the top the remaining sand in the top forms a smaller version of the cone that we started with. In particular the ratios of the radius and height of the current cone of sand is the same as the ratio of the radius and height of the original cone of sand. And so we have

$$\frac{r}{h} = \frac{2}{5} \quad \text{or} \quad r = \frac{2}{5}h.$$

(We used 2 since the diameter across is 4 inches so half of the diameter is the radius.) So substituting this in we now have the volume only as a function of h, i.e.,

$$V(h) = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h = \frac{4\pi}{75}h^3.$$

Taking the derivative of both sides with respect to t we have

$$\frac{dV}{dt} = \frac{4\pi}{75}\cdot 3h^2\frac{dh}{dt}.$$

Substituting what we know for $\frac{dV}{dt}$ as well as our desired $\frac{dh}{dt}$ we find

$$-2\pi = \frac{4\pi}{25}h^2(-2) \quad \text{or} \quad h^2 = \frac{25}{4} \quad \text{so} \quad h = \frac{5}{2} \text{ inches.}$$

4. (a) Find the three y-intercepts of the implicitly defined curve $y^3+4\sin(xy) = y+5x$.

A y-intercept occurs when we cross the y-axis, i.e., when $x = 0$. So plugging in $x = 0$ we have

$$y^3 = y \quad \text{or} \quad y^3 - y = 0 \quad \text{so} \quad y(y-1)(y+1) = 0.$$

So that the three y-intercepts are at $y = -1$, $y = 0$ and $y = 1$.

(b) For each point found in part (a) find the tangent line to the curve.

So the three points are $(0, -1)$, $(0, 0)$ and $(0, 1)$. We now just need to find the derivative and evaluate it at each of these points. So taking the derivative of both sides of the implicit relationship with respect to x we have

$$3y^2\frac{dy}{dx} + 4\cos(xy)\big(y + x\frac{dy}{dx}\big) = \frac{dy}{dx} + 5.$$

Now for all three of our points we have $x = 0$, so we might as well plug in $x = 0$ and then simplify, so that for our three y-intercepts we have

$$\big(3y^2 - 1\big)\frac{dy}{dx} = 5 - 4y \quad \text{or} \quad \frac{dy}{dx} = \frac{5-4y}{3y^2-1}.$$

So we have

$$\begin{aligned} \frac{dy}{dx}\bigg|_{(0,-1)} &= \frac{9}{2} \\ \frac{dy}{dx}\bigg|_{(0,0)} &= -5 \\ \frac{dy}{dx}\bigg|_{(0,1)} &= \frac{1}{2} \end{aligned}$$

5. (a) Find $\int x\big(\sin(x^2)+\cos(x^2)\big)^2\sin(x^2)\,dx$.

First let us make a substitution $u = x^2$ so that $du = 2x\,dx$ or $\frac{1}{2}\,du = x\,dx$ to rewrite this to get

$$\begin{aligned}\int x\big(\sin(x^2)+\cos(x^2)\big)^2\sin(x^2)\,dx &= \frac{1}{2}\int\big(\sin u+\cos u\big)^2\sin u\,du\\ &= \frac{1}{2}\int\big(\sin^2 u+2\sin u\cos u+\cos^2 u\big)\sin u\,du\\ &= \frac{1}{2}\int\big(1+2\sin u\cos u\big)\sin u\,du = \frac{1}{2}\int\sin u\,du+\int\underbrace{\sin^2 u\cos u\,du}_{\substack{v=\sin u\\ dv=\cos u\,du}}\\ &= -\frac{1}{2}\cos u+\int v^2\,dv = -\frac{1}{2}\cos u+\frac{1}{3}v^3+C = -\frac{1}{2}\cos u+\frac{1}{3}\sin^3 u+C\\ &= -\frac{1}{2}\cos(x^2)+\frac{1}{3}\sin^3(x^2)+C.\end{aligned}$$

Notice on this one we actually did substitution twice! There is no problem with repeated substitution, as long as we keep making the problem easier. Of course if we had a lot more foresight we could have gotten away with only making single substitutions, but usually we don't see that far ahead.

(b) Find $\int\frac{1-x^2}{1+x^2}\,dx$. (Hint: $1-x^2 = 2-(1+x^2)$.)

Following the hint we have

$$\begin{aligned}\int\frac{1-x^2}{1+x^2}\,dx = \int\frac{2-(1+x^2)}{1+x^2}\,dx &= \int\left(\frac{2}{1+x^2}-\frac{1+x^2}{1+x^2}\right)dx\\ &= \int\left(2\frac{1}{1+x^2}-1\right)dx = 2\arctan x - x + C.\end{aligned}$$

6. Given that $\int_3^x g(t)\,dt = \sqrt[3]{x^2-1} + Cx$, find C and $g(x)$.

First let us find C. One way to do this is to realize that for $x = 3$ we know the exact value of the right hand side, namely since we integrate from 3 to 3 the total integral is 0. So we have

$$0 = \sqrt[3]{3^2-1} + 3C = 2 + 3C \quad \text{so} \quad C = -\frac{2}{3}.$$

So now we have

$$\int_3^x g(t)\,dt = \sqrt[3]{x^2-1} - \frac{2}{3}x = (x^2-1)^{1/3} - \frac{2}{3}x,$$

taking the derivative of both sides and using the Fundamental Theorem of Calculus we have

$$g(x) = \frac{1}{3}(x^2-1)^{-2/3}\cdot(2x) - \frac{2}{3} = \frac{2x}{3(\sqrt[3]{x^2-1})^2} - \frac{2}{3}.$$

7. In the ten weeks leading up to the final you notice that the amount of time spent studying for this class has been increasing. In particular, you see that on the tth week of the quarter you were studying for $2 + t/5$ hours per week. Using integration, find the total amount of time that you spent studying for the final (i.e. from $t = 0$ weeks to $t = 10$ weeks).

To calculate the total we sum up the times which we can do via integration. Namely we have that the total time is

$$\int_0^{10} \big(2 + \frac{1}{5}t\big)\,dt = \big(2t + \frac{1}{10}t^2\big)\Big|_{t=0}^{t=10} = \big(20 + \frac{1}{10}(10)^2\big) - 0 = 20 + 10 = 30.$$

So a total of 34 hours studying for the test.

8. Find the average value for $f(x) = |4 - x^2|$ for the interval $-3 \le x \le 5$.

To find the average value we need to compute

$$\frac{1}{5-(-3)} \int_{-3}^{5} f(x) = \frac{1}{8} \int_{-3}^{5} f(x).$$

Since our function involves an absolute value, the easiest way to approach this will be to break it up into a piecewise function which does not have absolute values in it. To do this we look at when $4 - x^2 = 0$ which is at $x = \pm 2$, so we break the function up at these values so that

$$f(x) = |4 - x^2| = \begin{cases} x^2 - 4 & \text{if } x \le -2; \\ 4 - x^2 & \text{if } -2 \le x \le 2; \\ x^2 - 4 & \text{if } 2 \le x. \end{cases}$$

So to compute the integral we break it up to get the following.

$$\int_{-3}^{5} |4 - x^2|\,dx = \int_{-3}^{-2} (x^2 - 4)\,dx + \int_{-2}^{2} (4 - x^2)\,dx + \int_{2}^{5} (x^2 - 4)\,dx$$

$$= \big(\frac{1}{3}x^3 - 4x\big)\Big|_{x=-3}^{x=-2} + \big(4x - \frac{1}{3}x^3\big)\Big|_{x=-2}^{x=2} + \big(\frac{1}{3}x^3 - 4x\big)\Big|_{x=2}^{x=5}$$

$$= \big((-\frac{8}{3} + 8) - (-9 + 12)\big) + \big((8 - \frac{8}{3}) - (-8 + \frac{8}{3})\big) + \big((\frac{125}{3} - 20) - (\frac{8}{3} - 8)\big)$$

$$= \big(5 - \frac{8}{3}\big) + \big(16 - \frac{16}{3}\big) + \big(-12 + \frac{117}{3}\big) = 9 + \frac{93}{3} = 9 + 31 = 40.$$

So the average value is

$$\frac{1}{8} \int_{-3}^{5} f(x) = \frac{1}{8} \cdot 40 = 5.$$

9. (a) Use substition to show that for a function $f(x)$ that

$$\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx = \int_0^a \frac{f(a-x)}{f(x)+f(a-x)}\,dx.$$

If we make the substitution $u = a - x$, or $x = a - u$ then we have $du = -dx$ and so

$$\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx = -\int_a^0 \frac{f(a-u)}{f(a-u)+f(u)}\,du = \int_0^a \frac{f(a-u)}{f(a-u)+f(u)}\,du.$$

Since the u is a "dummy variable" we can replace all the u's in the last integral by x's and we get the result.

(b) Show that

$$\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx = \frac{1}{2}a.$$

(Hint: $\int_0^a g(x)\,dx = \frac{1}{2}\big(\int_0^a g(x)\,dx + \int_0^a g(x)\,dx\big)$, and use part (a).)

Using the hint we have (along with the previous part)

$$\begin{aligned}\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx &= \frac{1}{2}\left(\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx + \int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx\right)\\ &= \frac{1}{2}\left(\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx + \int_0^a \frac{f(a-x)}{f(x)+f(a-x)}\,dx\right)\\ &= \frac{1}{2}\int_0^a \frac{f(x)+f(a-x)}{f(x)+f(a-x)}\,dx\\ &= \frac{1}{2}\int_0^a 1\,dx = \frac{1}{2}a.\end{aligned}$$

(c) Find $\displaystyle\int_0^{\pi/2} \frac{1}{1+\tan x}\,dx$. (Hint: $\sin x = \cos\left(\frac{\pi}{2} - x\right)$.)

We first rewrite this and using the hint we have

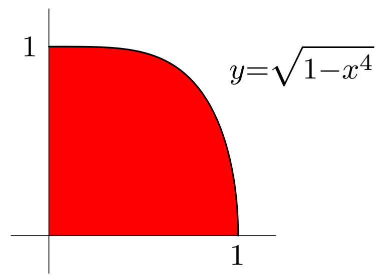
$$\int_0^{\pi/2} \frac{1}{1+\tan x}\,dx = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx = \int_0^{\pi/2} \frac{\cos x}{\cos x + \cos\left(\frac{\pi}{2} - x\right)}\,dx.$$

This is the integral done in part (b) where we have $a = \pi/2$ and $f(x) = \cos x$; so we can conclude that the integral is $\pi/4$.

10. Find the volume created when the region in the first quadrant bounded by the x-axis, the y-axis and $y = \sqrt{1 - x^4}$ is rotated

(a) around the x-axis.

First let us draw picture of exactly which region we are dealing with.



So to find the volume when rotating around the x-axis we have

$$\text{Volume} = \pi \int_0^1 \left(\sqrt{1 - x^4}\right)^2 dx = \pi \int_0^1 \left(1 - x^4\right) dx = \pi\left(x - \frac{1}{5}x^5\right)\Big|_{x=0}^{x=1}$$
$$= \pi\left(1 - \frac{1}{5}\right) - 0 = \frac{4}{5}\pi.$$

(b) around the y-axis. (Hint: $1 - x^4 = 1 - (x^2)^2$.)

Now we apply the formula to find the volume when rotating around the y-axis. We have

$$\text{Volume} = 2\pi \int_0^1 x\sqrt{1 - x^4}\, dx.$$

Now the integral isn't as nice as it was last time! Looking at our hint we see that we can express $1 - x^4$ as $1 - (x^2)^2$ and we also see that there is an $x\, dx$ on the outside of the square root. This suggests making the substitution $u = x^2$ and so $du = 2x\, dx$. While we are at it we can also change our bounds, in this case our bounds will now become $u = 0^2 = 0$ to $u = 1^2 = 1$ (the bounds are the same; but this will not always be the case!). And so we have

$$\text{Volume} = \pi \int_0^1 2x\sqrt{1 - (x^2)^2}\, dx = \pi \int_0^1 \sqrt{1 - u^2}\, du = \pi \cdot \frac{\pi}{4} = \frac{\pi^2}{4}.$$

In the last step we used the fact that $\int_0^1 \sqrt{1 - u^2}\, du$ corresponds to finding one-fourth of the area of a circle with radius 1, which is $\pi/4$.