MATH 31A (Butler)

Practice for Final (A)

Try to answer the following questions without the use of book, notes or calculator; but you can use the equation sheet posted on the course website. Time yourself and try to finish the questions in less than three hours.

1. Find $\displaystyle\int_{-\pi/2}^{\pi} \sin|\theta|\,d\theta$.

When we have an absolute value it is almost always best to break the function up into pieces. In this case for $\theta \le 0$ and $\theta \ge 0$. In particular we note that

$$\sin|\theta| = \begin{cases} \sin(-\theta) = -\sin\theta & \text{if } \theta \le 0, \\ \sin\theta & \text{if } \theta \ge 0. \end{cases}$$

So using this we now break up our integral and integrate each part. So we have

$$\begin{aligned}\int_{-\pi/2}^{\pi} \sin|\theta|\,d\theta &= \int_{-\pi/2}^{0} \sin|\theta|\,d\theta + \int_{0}^{\pi} \sin|\theta|\,d\theta \\ &= \int_{-\pi/2}^{0} (-\sin\theta)\,d\theta + \int_{0}^{\pi} \sin\theta\,d\theta = \cos\theta\Big|_{\theta=-\pi/2}^{\theta=0} - \cos\theta\Big|_{\theta=0}^{\theta=\pi} \\ &= \big(\cos 0 - \cos(-\tfrac{\pi}{2})\big) - \big(\cos\pi - \cos 0\big) = 3.\end{aligned}$$

2. (a) Find $\dfrac{d}{dx}\big(f(x)g(x)h(x)\big)$ in terms of $f(x)$, $g(x)$, $h(x)$, $f'(x)$, $g'(x)$ and $h'(x)$.

This is the product rule, but instead of two functions, now we have three functions! One way to think of this though is to group two of the functions together and then we apply the product rule twice. So we have

$$\begin{aligned}\frac{d}{dx}\big(f(x)g(x)h(x)\big) &= \frac{d}{dx}\Big(f(x)\big(g(x)h(x)\big)\Big) \\ &= f'(x)\big(g(x)h(x)\big) + f(x)\frac{d}{dx}\big(g(x)h(x)\big) \\ &= f'(x)\big(g(x)h(x)\big) + f(x)\big(g'(x)h(x) + g(x)h'(x)\big) \\ &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).\end{aligned}$$

(Note: this same pattern (where we add up all the ways to take the derivative of a single piece) holds in general for the product of any number of functions.)

(b) Starfleet intelligence has recently learned of a new threat. A new Borg vessel has been discovered that can change its shape, they are calling it the B-1000 (short for Borg-1000). The B-1000 though still has some limitations, first the only shape it can have is a three dimensional box and second the volume is always fixed, i.e., the box cannot deviate from a fixed volume.

You have been on a shuttle tracking the B-1000. From an earlier observation you saw that it had dimensions 20 meters by 15 meters by 10 meters. Currently though you can only see two sides. You notice that the length is currently 12 meters and is increasing at a rate of 1 meter per minute; the width is currently 25 meters and is decreasing at a rate of 2 meters per minute. What is the current depth of the B-1000 and how fast is it changing?

At first glance, this seems to have nothing to do with part (a), but let us continue and perhaps we will see the connection. First, we note that since the B-1000 is always a box that we have

$$V = xyz$$

where x, y and z are the length, width and depth respectively. The fact that the B-1000 cannot change its volume tells us that $\frac{dV}{dt} = 0$. From our earlier observation of the B-1000 we know that $V = 20{\cdot}15{\cdot}10 = 3000$. This allows us to easily solve for the depth since we have

$$3000 = 12{\cdot}25{\cdot}z = 300z \qquad \text{and so} \qquad z = 10.$$

To find how the depth is changing we take the derivative of both sides with respect to t. Ah, now we see why we did part (a), since xyz is the product of three functions and we are taking the derivative. So using part (a) we have

$$0 = \frac{dV}{dt} = \frac{dx}{dt}yz + x\frac{dy}{dt}z + xy\frac{dz}{dt}.$$

We know that $x = 12$ and that $\frac{dx}{dt} = 1$, $y = 25$ and $\frac{dy}{dt} = -2$, and from above that $z = 10$. Putting all of these in we have

$$0 = 1{\cdot}25{\cdot}10 + 12{\cdot}(-2){\cdot}10 + 12{\cdot}25{\cdot}\frac{dz}{dt} = 10 + 300\frac{dz}{dt},$$

and so

$$\frac{dz}{dt} = -\frac{10}{300} = -\frac{1}{30}.$$

So we see that the current depth of the B-1000 is 10 meters and is decreasing at a rate of $\frac{1}{30}$ meters per minute.

3. You and a classmate are preparing to give a presentation in your Astronomy 272 (stellar structure and evolution) course. You have decided that the best way to show how a star gets sucked into a black hole is through a modern interpretive dance where you will be playing the part of a large blue class O star and your partner will be the black hole. You will represent these two astronomical features using paper mache, and you are responsible for making your star. Initially you were planning to blow up a spherical balloon to a diameter of 16 inches before covering it in paper mache, but you ended up blowing the balloon to a diameter of 17 inches.

Using linear approximation get an estimate of how much more surface area the 17 inch balloon has as compared to the 16 inch balloon (i.e., an estimate of how much more paper you will need to make your model). (Hint: the surface area of a sphere of radius r is $4\pi r^2$.)

We have that

$$SA = 4\pi r^2,$$

and so using linear approximation we have

$$\Delta SA \approx 8\pi r \Delta r$$

(note that $8\pi r$ is the derivative of $4\pi r^2$). Since originally we had $r = 8$ (remember that r is half of the diameter) and now we moved r to $17/2$ (i.e., $\Delta r = 1/2$) we have

$$\Delta SA \approx 8\pi \cdot 8 \cdot \frac{1}{2} = 32\pi \text{ in}^2.$$

(Note: $32\pi \approx 100.53\ldots$, the actual difference is $\approx 103.67\ldots$; and so this means that our blue giant will require us to cover an additional area equivalent to a $10'' \times 10''$ square. All this from blowing up just one inch larger in diameter; that's the trouble you run into when you are full of hot air!)

4. Find the area of the largest rectangle that can be placed into the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. (Assume that the sides of the rectangle are parallel to the two axes.)

We can solve for y in terms of x. In particular we have

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2) \quad \text{or} \quad y = \pm\frac{b}{a}\sqrt{a^2 - x^2}.$$

In particular, the four corners of the rectangle will be located at

$$\big(\pm x, \pm\frac{b}{a}\sqrt{a^2 - x^2}\big).$$

The area of this rectangle will be

$$A(x) = \big(2x\big)\big(2\frac{b}{a}\sqrt{a^2 - x^2}\big) = \frac{4b}{a}x\sqrt{a^2 - x^2}.$$

To maximize we take the derivative and find the critical points (note that by the description of the problem that $0 \le x \le a$ but $x = 0$ and $x = a$ will have rectangles with area 0 and so will definitely not be maximal). So we have

$$\begin{aligned} A'(x) = \frac{4b}{a}\sqrt{a^2 - x^2} + \frac{4b}{a}x\frac{1}{2}\frac{1}{\sqrt{a^2 - x^2}}(-2x) &= \frac{4b}{a\sqrt{a^2 - x^2}}\big((a^2 - x^2) - x^2\big) \\ &= \frac{4b}{a\sqrt{a^2 - x^2}}\big(a^2 - 2x^2\big). \end{aligned}$$

At $x = a$ the derivative is undefined but we have already eliminated that possibility. To find the other critical points we set the numerator to 0 to get $2x^2 = a^2$ or $x = a/\sqrt{2}$. Since this is the only critical point then it must be the correct one, to be safe we note that for $x < a/\sqrt{2}$ the derivative is positive while if $x > a/\sqrt{2}$ the derivative is negative showing that $x = a/\sqrt{2}$ is a maximum.

Therefore the area of the largest rectangle is

$$A\big(\frac{a}{\sqrt{2}}\big) = \frac{4b}{a}\cdot\frac{a}{\sqrt{2}}\cdot\sqrt{a^2 - \big(\frac{a}{\sqrt{2}}\big)^2} = \frac{4b}{a}\cdot\frac{a}{\sqrt{2}}\cdot\frac{a}{\sqrt{2}} = 2ab.$$

5. Let $\ell(x) = \int_1^x \frac{1}{t}\,dt$.

(a) Find $\ell(1)$.

We have

$$\ell(1) = \int_1^1 \frac{1}{t}\,dt = 0,$$

since an integral that starts and stops at the same value is 0.

(b) Show that $\ell(ax) - \ell(x) = C$ where C is a constant. (Hint: a function is a constant if its derivative is 0.)

Following the hint we take the derivative of $\ell(ax) - \ell(x)$ using the Fundamental Theorem of Calculus to get

$$\frac{d}{dx}\big(\ell(ax) - \ell(x)\big) = \frac{1}{ax}\cdot(a) - \frac{1}{x} = \frac{1}{x} - \frac{1}{x} = 0.$$

Since the derivative is 0 then the function is a constant.

(c) Find the constant C for part (b). What can you conclude about $\ell(ax)$?

Since we have $\ell(ax) - \ell(x) = C$ for all x, let us pick a value that we can say something about. From part (a) we know $\ell(1) = 0$ and so putting in $x = 1$ we have

$$C = \ell(a) - \ell(1) = \ell(a).$$

Therefore we have that $\ell(ax) - \ell(x) = \ell(a)$, or rearranging,

$$\ell(ax) = \ell(a) + \ell(x).$$

(Note: the function $\ell(x)$ is actually the natural logarithm, which will be discussed in the next class. This shows that the logarithm turns multiplication into addition. This was the technology underlying the slide rulers, a useful tool for doing quick multiplication back in the dark ages before they had calculators, now they are almost nonexistent and have become museum pieces.)

6. Reduce the following to a single integral of the form $A\int_B^C f(x)\,dx$ for some constants A, B, C.

$$\int_0^5 f(x)\,dx - \int_3^3 f(x^2)\,dx + \int_0^1 3f(3x)\,dx - \int_0^4 f\big(\tfrac{1}{2}x\big)\,dx + \int_5^3 f(x)\,dx.$$

We first note that $\int_3^3 f(x^2)\,dx = 0$ since the ending and starting points are the same. We also have that $\int_5^3 f(x)\,dx = -\int_3^5 f(x)\,dx$ by a basic property of integrals. Using substitution ($u = 3x$ so $du = 3\,dx$) we have $\int_0^1 3f(3x)\,dx = \int_0^3 f(x)\,dx$, similarly we have $\int_0^4 f\big(\tfrac{1}{2}x\big)\,dx = 2\int_0^2 f(x)\,dx$. Putting all of this together we see that we can rewrite out expression as

$$\int_0^5 f(x)\,dx - 0 + \int_0^3 f(x)\,dx - 2\int_0^2 f(x)\,dx - \int_3^5 f(x)\,dx.$$

We can combine the first and last piece as follows

$$\int_0^5 f(x)\,dx - \int_3^5 f(x)\,dx = \int_0^3 f(x)\,dx$$

so that we now have

$$2\int_0^3 f(x)\,dx - 2\int_0^2 f(x)\,dx.$$

Finally, we can combine these two similarly as we did before to get

$$2\int_2^3 f(x)\,dx.$$

7. For $a \neq -3, -2, -1$ find $\int \left(1 + \sqrt[3]{x}\right)^a \, dx$. (Hint: try $u = 1 + \sqrt[3]{x}$.)

Let us start with the hint. If we have $u = 1 + \sqrt[3]{x} = 1 + x^{1/3}$ then we have

$$du = \frac{1}{3}x^{-2/3}\,dx \quad \text{or} \quad dx = 3x^{2/3}\,du = 3\left(\sqrt[3]{x}\right)^2 du.$$

But we want to be able to express du all in terms of u. Looking back at the original definition we note that we can solve for $\sqrt[3]{x} = u - 1$. So we have

$$du = 3\left(\sqrt[3]{x}\right)^2 du = 3(u-1)^2\,du.$$

Now we are ready to make the substitution.

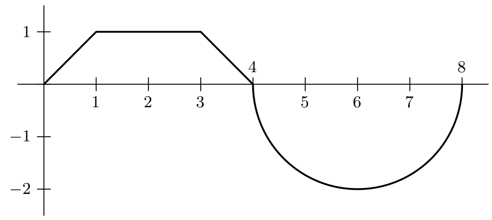
$$\begin{aligned}
\int \left(1 + \sqrt[3]{x}\right)^a dx &= \int u^a \cdot 3(u-1)^2\,du \\
&= 3\int (u^2 - 2u + 1)u^a\,du \\
&= 3\int (u^{2+a} - 2u^{1+a} + u^a)\,du
\end{aligned}$$

Note that since $a \neq -3, -2, -1$ that none of the exponents are -1 and so each of these integrals is easy.

$$\begin{aligned}
&= 3\left(\frac{1}{3+a}u^{3+a} - \frac{2}{2+a}u^{2+a} + \frac{1}{1+a}u^{1+a}\right) + C \\
&= 3\left(\frac{1}{3+a}\left(1+\sqrt[3]{x}\right)^{3+a} - \frac{2}{2+a}\left(1+\sqrt[3]{x}\right)^{2+a} + \frac{1}{1+a}\left(1+\sqrt[3]{x}\right)^{1+a}\right) + C
\end{aligned}$$

8. Let $g(x) = \int_2^x f(t)\,dt$ where $f(t)$ is the function defined piecewise by

$$f(t) = \begin{cases} t & \text{if } 0 \le t \le 1; \\ 1 & \text{if } 1 \le t \le 3; \\ 4-t & \text{if } 3 \le t \le 4; \\ -\sqrt{4-(t-6)^2} & \text{if } 4 \le t \le 8. \end{cases}$$



(a) Find the tangent line to $y = g(x)$ at $x = 6$.

To get the tangent line we need to find $g(6)$ and $g'(6)$. To find $g(6)$ we break the area up into pieces which are geometrical. So we have

$$g(6) = \int_2^6 f(t)\,dt = \int_2^3 f(t)\,dt + \int_3^4 f(t)\,dt + \int_4^6 f(t)\,dt$$
$$= 1 + \frac{1}{2} - \frac{1}{4}\pi 2^2 = \frac{3}{2} - \pi.$$

We also have

$$g'(6) = f(6) = -2.$$

Combining this we have

$$y = g(6) + g'(6)(x-6) = \left(\frac{3}{2} - \pi\right) - 2(x-6)$$

(b) There is exactly one critical point for $g(x)$ in the interval $0 < x < 8$. Find the x value of the critical point and classify it as a maximum, minimum or neither using the first derivative test.

Since $g'(x) = f(x)$, by looking at the graph provided we see that it is defined for the entire interval and the only point where it is 0 is at $x = 4$. So this is our critical point. We see that to the left of $x = 4$ that $g' > 0$ and to the right of $x = 4$ that $g' < 0$ so by the first derivative test we have that $x = 4$ is at a maximum.

(c) Could we have classified the critical point found in part (b) using the second derivative test? Explain (briefly).

No. The problem is that $g''(x) = f'(x)$ but at $x = 4$ the derivative of $f(x)$ is undefined (i.e., it is at a kink in the curve).

9. For $x \geq 1$ show

$$\arctan x - \frac{\pi}{4} \leq \int_1^x \frac{1}{t^2 + \cos^2 t}\,dt \leq 1 - \frac{1}{x}.$$

An important feature about $\cos^2 x$ is that it is strictly between 0 and 1. So we have

$$t^2 \leq t^2 + \cos^2 t \leq t^2 + 1 \quad \text{so} \quad \frac{1}{t^2+1} \leq \frac{1}{t^2 + \cos^2 t} \leq \frac{1}{t^2}.$$

Since we have inequalities between these functions we also have inequalities between the corresponding integrals, i.e.,

$$\int_1^x \frac{1}{t^2+1}\,dt \leq \int_1^x \frac{1}{t^2+\cos^2 t}\,dt \leq \int_1^x \frac{1}{t^2}\,dt.$$

We have

$$\int_1^x \frac{1}{t^2+1}\,dt = \arctan t\Big|_{t=1}^{t=x} = \arctan x - \arctan 1 = \arctan x - \frac{\pi}{4}$$

and

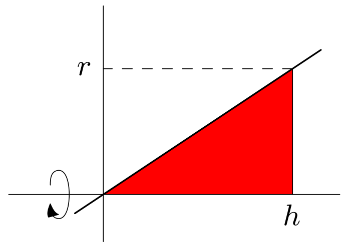
$$\int_1^x \frac{1}{t^2}\,dt = -\frac{1}{t}\Big|_{t=1}^{t=x} = -\frac{1}{x} - \left(-\frac{1}{1}\right) = 1 - \frac{1}{x}.$$

Combining everything together we have

$$\arctan x - \frac{\pi}{4} \leq \int_1^x \frac{1}{t^2 + \cos^2 t}\,dt \leq 1 - \frac{1}{x}.$$

10. Prove that a volume of a cone with radius r and height h is $\frac{1}{3}\pi r^2 h$. (Hint: one way to make such a cone is to take the triangle with vertices at $(0, 0)$, $(r, 0)$ and (r, h) and rotate it around the x-axis.)

To form the cone we rotate the region indicated by the hint around the x-axis, i.e., we have the following picture.



In particular, the region is bounded above by the line that passes through the points $(0, 0)$ and (h, r), which is the line $y = \frac{r}{h}x$, between $x = 0$ and $x = h$. Therefore the volume will be

$$\pi \int_0^h \left(\frac{r}{h}x\right)^2 dx = \frac{\pi r^2}{h^2} \int_0^h x^2\, dx = \frac{\pi r^2}{h^2} \cdot \frac{1}{3} x^3 \Big|_{x=0}^{x=h}$$

$$= \frac{\pi r^2}{h^2} \cdot \frac{1}{3} h^3 - \frac{\pi r^2}{h^2} \cdot \frac{1}{3} 0^3 = \frac{\pi r^2 h}{3}.$$