

# Discussion of “Duel”

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The argument below is a little more formal (and a little less direct) than the one we used in class, but the gist of the message is the same. Some of you may find it useful.

**The Rules.** There are two players, each with a wet sponge. They start at either side of the room. The players have alternate turns. Each turn, the player can either shoot or take a step forward. If she shoots and hits then she wins. If she misses, then the game continues so effectively she loses (since the other player can wait and shoot point blank).

**The Strategic Decision.** When to shoot?

**Extra structure to make the analysis tractable.** Assume that the abilities of the players are known. In particular,

- let  $P_1(d)$  be player 1’s probability of hitting if she shoots at distance  $d$ ; and
- let  $P_2(d)$  be player 2’s probability of hitting if she shoots at distance  $d$ ;

In addition, let us make two plausible assumptions:

- $P_1(0) = P_2(0) = 1$ ; that is, they hit for sure from point blank.
- Both  $P_1$  and  $P_2$  are decreasing; that is, the probabilities of hitting go down with distance.

Notice we do need not assume that the players are equally good shots, or even that one is better than the other at all distances, just that their abilities are known.

**Speculations.** We might think that a better shot will shoot first since she has a better chance of hitting. Alternatively, we might think that a worse shot will try to pre-empt the better shot. In fact, the right way to solve this game is: *backward induction*.

First, let us establish some preliminary observations that will help us.

**Observation A.** *If player  $i$  knows ‘today’ (at  $d$ ) that player  $j$  **won’t** shoot ‘tomorrow’ (at  $d - 1$ ) then player  $i$  should not shoot today: she does better by waiting to shoot (say) the day after tomorrow (at  $d - 2$ ).*

**Observation B.** *If player  $i$  knows ‘today’ (at  $d$ ) that player  $j$  **will** shoot ‘tomorrow’ (at  $d - 1$ ) then player  $i$  should shoot today if and only if the probability of  $i$  winning by hitting today,  $P_i(d)$ , is higher than the probability of  $i$  winning by  $j$  missing tomorrow,  $[1 - P_j(d - 1)]$ . That is, shoot if  $P_i(d) > [1 - P_j(d - 1)]$ ; or equivalently, shoot if*

$$P_i(d) + P_j(d - 1) > 1$$

**THE SOLUTION** I claim that the *solution* — the way to play this game — is to shoot the first time that the inequality above holds. To see this, we proceed by our favorite method.

**Backward Induction.** Start at the last possible decision node; that is, when they are standing — perhaps foolishly — “nose to nose”.

**At  $d = 0$ .** (Suppose it is player 2’s turn.) Player 2 *will* shoot since  $P_2(0) = 1$ .

**At  $d = 1$ .** (Player 1’s turn.) By the  $d = 0$  argument, player 1 knows player 2 *will* shoot tomorrow. Therefore, by observation **B** above, player 1 should shoot if and only if  $P_1(1) > 1 - P_2(0)$ , which is true since  $P_2(0) = 1$ . So player 1 *will* shoot at  $d = 1$ .

**At  $d = 2$ .** (Player 2’s turn.) By the  $d = 1$  argument, player 2 knows player 1 *will* shoot tomorrow. Therefore, by observation **B** above, player 2 should shoot if and only if:

$$P_2(2) + P_1(1) \stackrel{?}{>} 1. \tag{*}$$

This inequality may be true or false. If it is false, then we are done: at  $d = 2$ , player 2 *won’t* shoot. At  $d = 3$ , player 1 knows that player 2 *won’t* shoot ‘tomorrow’ (at  $d = 2$ ). So, by observation **A** above, player 1 will not shoot ‘today’ (at  $d = 3$ ). Similarly, at  $d = 4$ , player 2 knows that player 1 *won’t* shoot at  $d = 3$ . So player 2 *won’t* shoot at  $d = 4$ . . . . Thus, in this case, the first shot is fired at  $d = 1$ .

Suppose then that inequality (\*) is true. In this case, player 2 will shoot at  $d = 2$ , so let’s consider  $d = 3$ .

**At  $d = 3$ .** (Player 1’s turn.) Since we are assuming that inequality (\*) is true, by the  $d = 2$  argument, player 1 knows that player 2 *will* shoot tomorrow. In this case, by observation **B** above, player 1 should shoot if and only if  $P_1(3) > 1 - P_2(2)$  or equivalently:

$$P_1(3) + P_2(2) \stackrel{?}{>} 1. \tag{**}$$

This inequality may be true or false. If it is false, using observation **A** as before, we are done: in this case, the first shot is fired at  $d = 2$ .

If not, we can continue the argument backward, moving our duelists further and further apart, and generating inequalities like (\*) and (\*\*). Notice that the left side of these “star”-inequalities gets smaller and smaller as we move further apart. As above, as soon as a “star”-inequality fails we are done. Therefore let  $d^*$  be the largest distance such that the “star”-inequality still holds. That is,

$$\begin{aligned} P_i(d^*) + P_j(d^* - 1) &> 1 \text{ but} \\ P_j(d^* + 1) + P_i(d^*) &< 1, \end{aligned}$$

where  $i$  is the player whose turn it is at  $d^*$ . By our earlier argument, we know that the first shot occurs at  $d^*$ . ■

Now that we have analyzed the game backward, let’s just rephrase things forward. The players step forward until the first distance such that the “star”-inequality holds. This distance

is  $d^*$ . Whoever's turn it is that point shoots. Notice that it does not matter whether you are the better or worse shot — it is the sum of the two hitting-probabilities that matters.

We can get a tidier answer if we notice that  $d^*$  is very close to being the point where  $P_i(d) + P_j(d) = 1$ . Our rule is then “shoot as soon as you cross the point where the sum of the probabilities of hitting equals one”!

One thing may still be worrying you. how does the agent at  $d^*$  know that the agent tomorrow will shoot if she does not? The answer is that, since the “star”-inequality holds at  $d^*$  and since the left side of the inequalities is decreasing in distance, the star-inequality will continue to hold at all closer (subsequent) distances. So by the same backward-induction argument that we used at  $d = 1$ ,  $d = 2$  and  $d = 3$ , the agent should shoot.

**Lesson 1.** *In many games, the decision that matters is not what to do but when to do it. A related example is when to launch a product when: you know that only one product will survive; a rival is also developing a product; and if you launch an imperfect product too early and it fails, your reputation will be “shot”.*

**Lesson 2.** *This game is hard but, if we keep our heads, backward induction yields a simple solution: roughly, regardless of whether you are the better or worse shot, **shoot when the sum of the hit-probabilities is equal to one.***