Problem Set 10

Econ 159a/MGT522a   Ben Polak

Four Problems Due December 5, 2007

1. Using different equilibria to create incentives. Alex and Barry have a joint project. Each has first to decide whether to invest $10 or zero (i.e., not to invest) into the project. They make these individual investment choices simultaneously. Once made, these investments are sunk. If no-one invests, the project generates a total revenue of $0. If just one of them invests, then the project generates a (gross) total revenue of $15. If both of them invests, the project generates a (gross) total revenue of $30.

Alex and Barry then use the following scheme to divide the total project revenue. Each player simultaneously writes down a 'share demand' on a piece of paper. The demands can be either $\frac{1}{5},\frac{1}{2}$, or $\frac{4}{5}$. If the two 'share demands' add up exactly to one then each player is given his demand. Otherwise, all the money is thrown away.

[For example, if at first Alex and Barry each invest $10 then the project generates a gross total of $30. If Alex then writes down $\frac{4}{5}$ and Barry writes down $\frac{1}{5}$ then (since this adds up to one) Alex gets his demand of $24 (= $\frac{4}{5} \times$ $30) for a net profit of $14 (i.e., $24 minus his initial investment of $10), while Barry gets his demand of $6 (= $\frac{1}{5} \times$ $30) for a net profit of $-$4 (i.e., $6 minus his initial investment of $10). If Barry had demanded $\frac{1}{2}$ while Alex was still demanding $\frac{4}{5}$, then the project money would have been thrown away and each would simply have lost his initial investment of $10.]

Suppose for now that both Alex and Barry are told how much the project has generated before they make their 'share demands'.

(a) Consider the subgame that follows Alex choosing to invest $10 and Barry choosing to invest $0. Find all the pure-strategy NE of this subgame.

(b) Is there a pure-strategy sub-game perfect equilibrium (SPE) of the whole game in which each player starts by investing $0? If so, explain the equilibrium strategies clearly. If not, explain why not clearly.

(c) Is there a pure-strategy SPE in which each player starts by investing $10? If so, explain the equilibrium strategies clearly. If not, explain why not clearly.

Suppose now that, at the point at which each player has to make his 'share demand', he does not know what amount the other player has invested in the project and hence does not know how much the project has generated.

(d) [**6 points**] Is there now a pure-strategy SPE in which each player starts by investing $10. If so, explain the equilibrium strategies clearly. If not, explain why not.

2. A finitely repeated game. Consider the two-player game

	a	b	c	d
A	3,1	0,0	0,0	5,0
B	0,0	1,3	0,0	0,0
C	0,0	0,0	2,2	0,0
D	0,0	0,5	0,0	4,4

(a) Find all the pure-strategy Nash equilibria of this game.

(b) Suppose that this game is played twice (i.e., played and then repeated once). Construct a pure-strategy SPE in which (D, d) is played in the first stage.

(c) Argue briefly that this SPE is more robust to the problem of renegotiation than is the equilibrium in the two-stage game we discussed in class.

3. Infinitely repeated games: an application. Heated Competition. *New Haven Heat* and *New Haven Warmth* are the only two firms allowed to provide home-heating oil in New Haven. Each firm has a constant marginal cost of supplying oil equal to $1 per gallon. Let the prices per gallon of the two firms be p_h and p_w respectively. Heating oil is a perfectly homogeneous good, so all customers buy from whichever company offers the lower price. The total demand for oil in New Haven is given by the following demand function:

$$Q(p_L) = 200,000 - 100,000p_L \text{ gallons,}$$

where p_L is whichever is the lower of p_h and p_w. For example, if $p_h = \$0$ and $p_w = \$1.50$, then total sales in New Haven would be $200,000$ gallons, all customers would buy from *Heat*, and *Heat* would make losses of $200,000. If $p_h = \$1.75$ and $p_w = \$1.25$, then total sales would be $200,000 - 100,000 \, (1.25) = 75,000$ gallons, all customers would buy from *Warmth*, and *Warmth* would make profits of $18,750. Assume that, if *Heat* and *Warmth* announce the same price, demand splits exactly equally between the two firms.

(a) For the moment, suppose that this competition between *Heat* and *Warmth* occurs just once. Suppose that the firms announce their prices simultaneously.

   (i) What prices are strictly dominated strategies, and what prices are weakly dominated strategies?

   (ii) Find all the Nash equilibria in this game. For all the Nash equilibria in this game, what is the equilibrium price p_L.

(b) Now suppose that this competition is played repeatedly, year after year, and that both firms have discount factor δ.

   (i) Find the lowest δ such that the firms are able to sustain the monopoly price in a subgame-perfect equilibrium. Construct such an equilibrium and explain briefly why no other subgame-perfect equilibrium can sustain the monopoly price at a lower δ.

(ii) Suppose instead that the demand for heating is given by $Q(p_L) = a - bp_L$ gallons, where $a > 1$, and $b > 0$. In this case, what is the lowest δ such that firms can sustain the monopoly price in a subgame-perfect equilibrium. Explain what is general about this result and why?

4. Infinitely repeated games: practice at the theory. Consider the following game.

	c	d
c	$2, 1$	$-1, 4$
d	$3, -3$	$0, 0$

Assume that this game is repeated an infinite number of times, and that both the row and column player discount the future with the same discount factor δ.

(a) Suppose that both players follow the following grim-trigger strategy: "play c as long as no-one has ever played d; otherwise play d". Find the minimum value of δ such that this is a subgame-perfect equilibrium.

(b) Suppose that row and column are playing *some* SPE equilibrium of the infinitely repeated game. They may or may not have played according to the equilibrium strategies so far. Let V^r and V^c denote the present discounted values of continuing to play from here on according to the equilibrium strategies. What are the lowest values that V^r and V^c could have? [Hint: there are no calculations involved here.]

(c) [Optional.] Now suppose that the players follow a strategy in which they both start in "phase C" (described below) and then switch "phases" as the instructions dictate:

- phase C: play c provided that, in each period since the (re)start of the phase, either both players chose c or both players chose d. Stay in phase C until one player chooses d while the other chooses c. In this event, if it is row who chose d go to phase P_r; if it is column who chose d go to phase P_c.

- phase P_r: play d for T_r periods (regardless of what happens during those periods) then revert to phase C.

- phase P_c: play d for T_c periods (regardless of what happens during those periods) then revert to phase C.

Write down expressions in terms of δ for the smallest T_r and T_c such that this is an SPE. [These expressions might be quite ugly so do not attempt to simplify them much.] For any given δ, which of T_r and T_c is larger? What is the intuition?

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