

## Introduction to Quantum Mechanics HT 2010

### Problems 5 (week 3)

**5.1** Show that  $L_i$  commutes with  $\mathbf{x} \cdot \mathbf{p}$  and thus also with scalar functions of  $\mathbf{x}$  and  $\mathbf{p}$ .

**5.2** In the rotation spectrum of  $^{12}\text{C}^{16}\text{O}$  the line arising from the transition  $l = 4 \rightarrow 3$  is at 461.04077 GHz, while that arising from  $l = 36 \rightarrow 35$  is at 4115.6055 GHz. Show from these data that in a non-rotating CO molecule the intra-nuclear distance is  $s \simeq 0.113$  nm, and that the electrons provide a spring between the nuclei that has force constant  $\sim 1904$  N m $^{-1}$ . Hence show that the vibrational frequency of CO should lie near  $6.47 \times 10^{13}$  Hz (measured value is  $6.43 \times 10^{13}$  Hz). Hint: show from classical mechanics that the distance of O from the centre of mass is  $\frac{3}{7}s$  and that the molecule's moment of inertia is  $\frac{48}{7}m_{\text{p}}s^2$ . Recall also the classical relation  $L = I\omega$ .

**5.3** The angular part of a system's wavefunction is

$$\langle \theta, \phi | \psi \rangle \propto (\sqrt{2} \cos \theta + \sin \theta e^{-i\phi} - \sin \theta e^{i\phi}).$$

What are the possible results of measurement of (a)  $L^2$ , and (b)  $L_z$ , and their probabilities? What is the expectation value of  $L_z$ ?

**5.4** A system's wavefunction is proportional to  $\sin^2 \theta e^{2i\phi}$ . What are the possible results of measurements of (a)  $L_z$  and (b)  $L^2$ ?

**5.5** A system's wavefunction is proportional to  $\sin^2 \theta$ . What are the possible results of measurements of (a)  $L_z$  and (b)  $L^2$ ? Give the probabilities of each possible outcome.

**5.6** Let  $\mathbf{n}$  be any unit vector and  $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$  be the vector whose components are the Pauli matrices. Why is it physically necessary that  $\mathbf{n} \cdot \boldsymbol{\sigma}$  satisfy  $(\mathbf{n} \cdot \boldsymbol{\sigma})^2 = I$ , where  $I$  is the  $2 \times 2$  identity matrix? Let  $\mathbf{m}$  be a unit vector such that  $\mathbf{m} \cdot \mathbf{n} = 0$ . Why do we require that the commutator  $[\mathbf{m} \cdot \boldsymbol{\sigma}, \mathbf{n} \cdot \boldsymbol{\sigma}] = 2i(\mathbf{m} \times \mathbf{n}) \cdot \boldsymbol{\sigma}$ ? Prove that these relations follow from the algebraic properties of the Pauli matrices. You should be able to show that  $[\mathbf{m} \cdot \boldsymbol{\sigma}, \mathbf{n} \cdot \boldsymbol{\sigma}] = 2i(\mathbf{m} \times \mathbf{n}) \cdot \boldsymbol{\sigma}$  for any two vectors  $\mathbf{n}$  and  $\mathbf{m}$ .

**5.7** Let  $\mathbf{n}$  be the unit vector in the direction with polar coordinates  $(\theta, \phi)$ . Write down the matrix  $\mathbf{n} \cdot \boldsymbol{\sigma}$  and find its eigenvectors. Hence show that the state of a spin-half particle in which a measurement of the component of spin along  $\mathbf{n}$  is certain to yield  $\frac{1}{2}\hbar$  is

$$|+, \mathbf{n}\rangle = \sin(\theta/2) e^{i\phi/2} |-\rangle + \cos(\theta/2) e^{-i\phi/2} |+\rangle, \quad (5.1)$$

where  $|\pm\rangle$  are the states in which  $\pm\frac{1}{2}$  is obtained when  $s_z$  is measured. Obtain the corresponding expression for  $|-, \mathbf{n}\rangle$ . Explain physically why the amplitudes in (5.1) have modulus  $2^{-1/2}$  when  $\theta = \pi/2$  and why one of the amplitudes vanishes when  $\theta = \pi$ .

**5.8** Write down the  $3 \times 3$  matrix that represents  $S_x$  for a spin-one system in the basis in which  $S_z$  is diagonal (i.e., the basis states are  $|0\rangle$  and  $|\pm\rangle$  with  $S_z|+\rangle = |+\rangle$ , etc.)

A beam of spin-one particles emerges from an oven and enters a Stern–Gerlach filter that passes only particles with  $J_z = \hbar$ . On exiting this filter, the beam enters a second filter that passes only particles with  $J_x = \hbar$ , and then finally it encounters a filter that passes only particles with  $J_z = -\hbar$ . What fraction of the particles stagger right through?

**5.9** A box containing two spin-one gyros A and B is found to have angular-momentum quantum numbers  $j = 2$ ,  $m = 1$ . Determine the probabilities that when  $J_z$  is measured for gyro A, the values  $m = \pm 1$  and 0 will be obtained.

What is the value of the Clebsch–Gordan coefficient  $C(2, 1; 1, 1, 1, 0)$ ?

**5.10** The angular momentum of a hydrogen atom in its ground state is entirely due to the spins of the electron and proton. The atom is in the state  $|1, 0\rangle$  in which it has one unit of angular momentum but none of it is parallel to the  $z$ -axis. Express this state as a linear combination of products of the spin states  $|\pm, e\rangle$  and  $|\pm, p\rangle$  of the proton and electron. Show that the states  $|x\pm, e\rangle$  in which the electron has well-defined spin along the  $x$ -axis are

$$|x\pm, e\rangle = \frac{1}{\sqrt{2}} (|+, e\rangle \pm |-, e\rangle). \quad (5.2)$$

By writing

$$|1, 0\rangle = |x+, e\rangle \langle x+, e|1, 0\rangle + |x-, e\rangle \langle x-, e|1, 0\rangle, \quad (5.3)$$

express  $|1, 0\rangle$  as a linear combination of the products  $|x\pm, e\rangle |x\pm, p\rangle$ . Explain the physical significance of your result.