Modules 31 and 32: Interference and Diffraction[1]

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[1] These notes are excerpted "Introduction to Electricity and Magnetism" by Sen-Ben Liao, Peter Dourmashkin, and John Belcher, Copyright 2004, ISBN 0-536-81207-1.

Interference and Diffraction

14.1 Superposition of Waves

Consider a region in space where two or more waves pass through at the same time. According to the superposition principle, the net displacement is simply given by the vector or the algebraic sum of the individual displacements. Interference is the combination of two or more waves to form a composite wave, based on such principle. The idea of the superposition principle is illustrated in Figure 14.1.1.

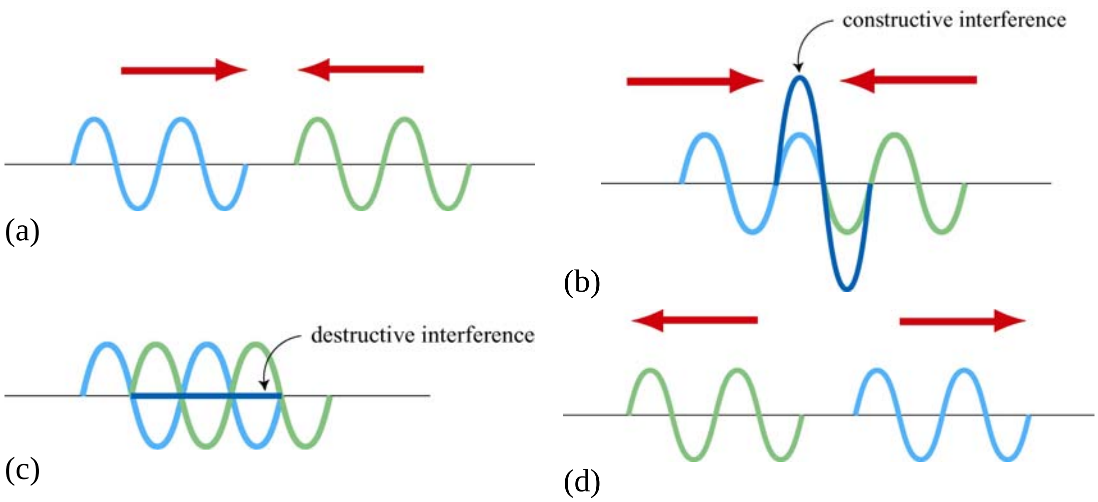


Figure 14.1.1 Superposition of waves. (b) Constructive interference, and (c) destructive interference.

Suppose we are given two waves,

$$\psi_1(x,t) = \psi_{10}\sin(k_1 x \pm \omega_1 t + \phi_1), \qquad \psi_2(x,t) = \psi_{20}\sin(k_2 x \pm \omega_2 t + \phi_2) \tag{14.1.1}$$

the resulting wave is simply

$$\psi(x,t) = \psi_{10}\sin(k_1 x \pm \omega_1 t + \phi_1) + \psi_{20}\sin(k_2 x \pm \omega_2 t + \phi_2) \tag{14.1.2}$$

The interference is constructive if the amplitude of $\psi(x,t)$ is greater than the individual ones (Figure 14.1.1b), and destructive if smaller (Figure 14.1.1c).

As an example, consider the superposition of the following two waves at $t = 0$:

$$\psi_1(x) = \sin x, \qquad \psi_2(x) = 2\sin\left(x + \frac{\pi}{4}\right) \tag{14.1.3}$$

The resultant wave is given by

$$\psi(x)=\psi_1(x)+\psi_2(x)=\sin x+2\sin\left(x+\frac{\pi}{4}\right)=\left(1+\sqrt{2}\right)\sin x+\sqrt{2}\cos x \tag{14.1.4}$$

where we have used

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta \tag{14.1.5}$$

and $\sin(\pi/4)=\cos(\pi/4)=\sqrt{2}/2$. Further use of the identity

$$\begin{aligned} a\sin x+b\cos x &= \sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\sin x+\frac{b}{\sqrt{a^2+b^2}}\cos x\right] \\ &= \sqrt{a^2+b^2}\left[\cos\phi\,\sin x+\sin\phi\,\cos x\right] \\ &= \sqrt{a^2+b^2}\,\sin(x+\phi) \end{aligned} \tag{14.1.6}$$

with

$$\phi=\tan^{-1}\left(\frac{b}{a}\right) \tag{14.1.7}$$

then leads to

$$\psi(x)=\sqrt{5+2\sqrt{2}}\,\sin(x+\phi) \tag{14.1.8}$$

where $\phi=\tan^{-1}(\sqrt{2}/(1+\sqrt{2}))=30.4°=0.53\text{ rad}$. The superposition of the waves is depicted in Figure 14.1.2.

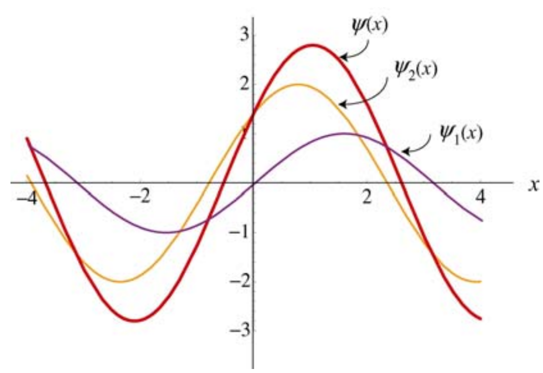


Figure 14.1.2 Superposition of two sinusoidal waves.

We see that the wave has a maximum amplitude when $\sin(x+\phi)=1$, or $x=\pi/2-\phi$. The interference there is constructive. On the other hand, destructive interference occurs at $x=\pi-\phi=2.61\text{ rad}$, where $\sin(\pi)=0$.

In order to form an interference pattern, the incident light must satisfy two conditions:

(i) The light sources must be *coherent*. This means that the plane waves from the sources must maintain a constant phase relation. For example, if two waves are completely out of phase with $\phi = \pi$, this phase difference must not change with time.

(ii) The light must be *monochromatic*. This means that the light consists of just one wavelength $\lambda = 2\pi / k$.

Light emitted from an incandescent lightbulb is *incoherent* because the light consists o waves of different wavelengths and they do not maintain a constant phase relationship. Thus, no interference pattern is observed.

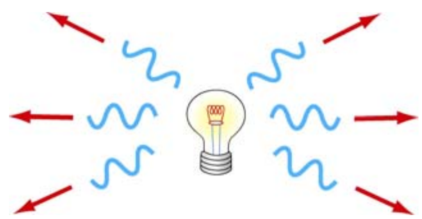


Figure 14.1.3 Incoherent light source

14.2 Young's Double-Slit Experiment

In 1801 Thomas Young carried out an experiment in which the wave nature of light was demonstrated. The schematic diagram of the double-slit experiment is shown in Figure 14.2.1.

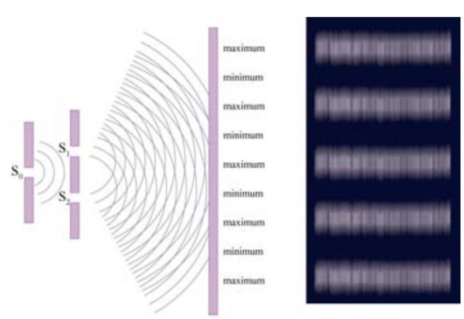


Figure 14.2.1 Young's double-slit experiment.

A monochromatic light source is incident on the first screen which contains a slit S_0. The emerging light then arrives at the second screen which has two parallel slits S_1 and S_2. which serve as the sources of coherent light. The light waves emerging from the two slits then interfere and form an interference pattern on the viewing screen. The bright bands (fringes) correspond to interference maxima, and the dark band interference minima.

Figure 14.2.2 shows the ways in which the waves could combine to interfere constructively or destructively.

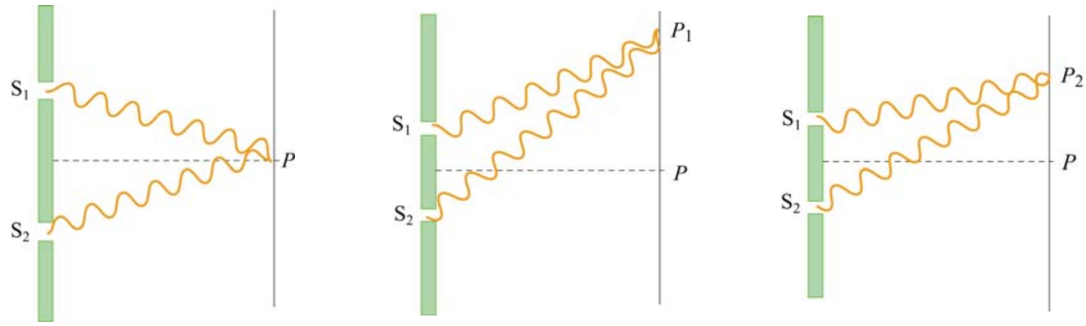


Figure 14.2.2 Constructive interference (a) at *P*, and (b) at P_1. (c) Destructive interference at P_2.

The geometry of the double-slit interference is shown in the Figure 14.2.3.

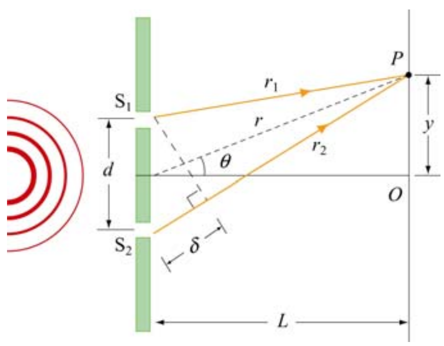


Figure 14.2.3 Double-slit experiment

Consider light that falls on the screen at a point P a distance y from the point O that lies on the screen a perpendicular distance L from the double-slit system. The two slits are separated by a distance *d*. The light from slit 2 will travel an extra distance $\delta = r_2 - r_1$ to the point P than the light from slit 1. This extra distance is called the *path difference*. From Figure 14.2.3, we have, using the law of cosines,

$$r_1^2 = r^2 + \left(\frac{d}{2}\right)^2 - dr\cos\left(\frac{\pi}{2} - \theta\right) = r^2 + \left(\frac{d}{2}\right)^2 - dr\sin\theta \qquad (14.2.1)$$

and

$$r_2^2 = r^2 + \left(\frac{d}{2}\right)^2 - dr\cos\left(\frac{\pi}{2} + \theta\right) = r^2 + \left(\frac{d}{2}\right)^2 + dr\sin\theta \qquad (14.2.2)$$

Subtracting Eq. (142.1) from Eq. (14.2.2) yields

$$r_2^{\,2} - r_1^2 = (r_2 + r_1)(r_2 - r_1) = 2dr\sin\theta \qquad (14.2.3)$$

In the limit $L \gg d$, i.e., the distance to the screen is much greater than the distance between the slits, the sum of r_1 and r_2 may be approximated by $r_1 + r_2 \approx 2r$, and the path difference becomes

$$\delta = r_2 - r_1 \approx d\sin\theta \qquad (14.2.4)$$

In this limit, the two rays r_1 and r_2 are essentially treated as being parallel (see Figure 14.2.4).

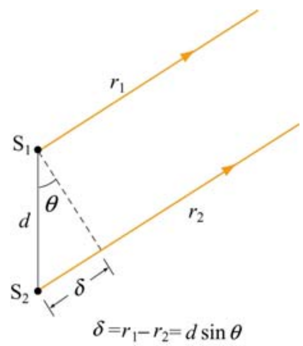


Figure 14.2.4 Path difference between the two rays, assuming $L \gg d$.

Whether the two waves are in phase or out of phase is determined by the value of δ. Constructive interference occurs when δ is zero or an integer multiple of the wavelength λ:

$$\boxed{\delta = d\sin\theta = m\lambda, \quad m = 0, \pm 1, \pm 2, \pm 3, \ldots \text{ (constructive interference)}} \qquad (14.2.5)$$

where m is called the *order number*. The zeroth-order (m = 0) maximum corresponds to the central bright fringe at $\theta = 0$, and the first-order maxima ($m = \pm 1$) are the bright fringes on either side of the central fringe.

On the other hand, when δ is equal to an odd integer multiple of $\lambda / 2$, the waves will be $180°$ out of phase at *P*, resulting in destructive interference with a dark fringe on the screen. The condition for destructive interference is given by

$$\boxed{\delta = d\sin\theta = \left(m + \frac{1}{2}\right)\lambda, \quad m = 0, \pm 1, \pm 2, \pm 3, \ldots \text{ (destructive interference)}} \qquad (14.2.6)$$

In Figure 14.2.5, we show how a path difference of $\delta = \lambda / 2$ ($m = 0$) results in a destructive interference and $\delta = \lambda$ ($m = 1$) leads to a constructive interference.

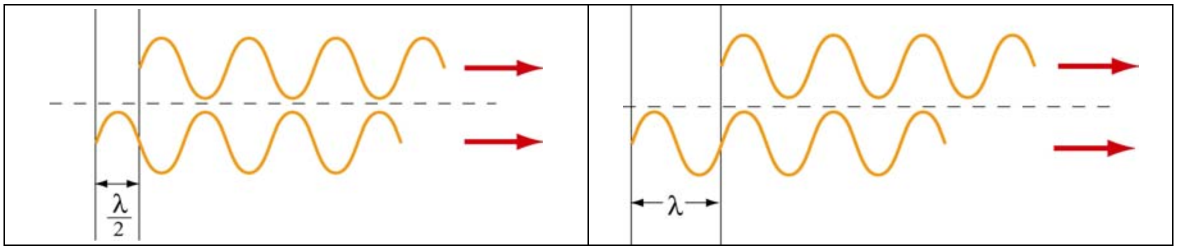


Figure 14.2.5 (a) Destructive interference. (b) Constructive interference.

To locate the positions of the fringes as measured vertically from the central point *O*, in addition to $L \gg d$, we shall also assume that the distance between the slits is much greater than the wavelength of the monochromatic light, $d \gg \lambda$. The conditions imply that the angle θ is very small, so that

$$\sin\theta \approx \tan\theta = \frac{y}{L} \tag{14.2.7}$$

Substituting the above expression into the constructive and destructive interference conditions given in Eqs. (14.2.5) and (14.2.6), the positions of the bright and dark fringes are, respectively,

$$y_b = m\frac{\lambda L}{d} \tag{14.2.8}$$

and

$$y_d = \left(m + \frac{1}{2}\right)\frac{\lambda L}{d} \tag{14.2.9}$$

Example 14.1: Double-Slit Experiment

Suppose in the double-slit arrangement, $d = 0.150\,\text{mm}$, $L = 120\,\text{cm}$, $\lambda = 833\,\text{nm}$, and $y = 2.00\,\text{cm}$.

(a) What is the path difference δ for the rays from the two slits arriving at point *P*?

(b) Express this path difference in terms of λ.

(c) Does point *P* correspond to a maximum, a minimum, or an intermediate condition?

Solutions:

(a) The path difference is given by $\delta = d\sin\theta$. When $L \gg y$, θ is small and we can make the approximation $\sin\theta \approx \tan\theta = y/L$. Thus,

$$\delta \approx d\left(\frac{y}{L}\right) = \left(1.50\times10^{-4}\ \mathrm{m}\right)\frac{2.00\times10^{-2}\ \mathrm{m}}{1.20\,\mathrm{m}} = 2.50\times10^{-6}\ \mathrm{m}$$

(b) From the answer in part (a), we have

$$\frac{\delta}{\lambda} = \frac{2.50\times10^{-6}\ \mathrm{m}}{8.33\times10^{-7}\ \mathrm{m}} \approx 3.00$$

or $\delta = 3.00\lambda$.

(c) Since the path difference is an integer multiple of the wavelength, the intensity at point P is a maximum.

14.3 Intensity Distribution

Consider the double-slit experiment shown in Figure 14.3.1.

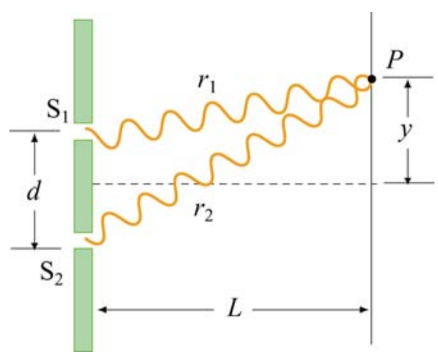


Figure 14.3.1 Double-slit interference

The total instantaneous electric field $\vec{\mathbf{E}}$ at the point P on the screen is equal to the vector sum of the two sources: $\vec{\mathbf{E}} = \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2$. On the other hand, the Poynting flux S is proportional to the square of the total field:

$$S \propto E^2 = (\vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2)^2 = E_1^2 + E_2^2 + 2\vec{\mathbf{E}}_1 \cdot \vec{\mathbf{E}}_2 \quad (14.3.1)$$

Taking the time average of S, the intensity I of the light at P may be obtained as:

$$I = \langle S \rangle \propto \langle E_1^2 \rangle + \langle E_2^2 \rangle + 2\langle \vec{\mathbf{E}}_1 \cdot \vec{\mathbf{E}}_2 \rangle \quad (14.3.2)$$

The cross term $2\langle \vec{\mathbf{E}}_1 \cdot \vec{\mathbf{E}}_2 \rangle$ represents the correlation between the two light waves. For incoherent light sources, since there is no definite phase relation between $\vec{\mathbf{E}}_1$ and $\vec{\mathbf{E}}_2$, the cross term vanishes, and the intensity due to the incoherent source is simply the sum of the two individual intensities:

$$I_{\text{inc}} = I_1 + I_2 \tag{14.3.3}$$

For coherent sources, the cross term is non-zero. In fact, for constructive interference, $\vec{\mathbf{E}}_1 = \vec{\mathbf{E}}_2$, and the resulting intensity is

$$I = 4I_1 \tag{14.3.4}$$

which is four times greater than the intensity due to a single source. On the other hand, when destructive interference takes place, $\vec{\mathbf{E}}_1 = -\vec{\mathbf{E}}_2$, and $\langle \vec{\mathbf{E}}_1 \cdot \vec{\mathbf{E}}_2 \rangle \propto -I_1$, and the total intensity becomes

$$I = I_1 - 2I_1 + I_1 = 0 \tag{14.3.5}$$

as expected.

Suppose that the waves emerged from the slits are coherent sinusoidal plane waves. Let the electric field components of the wave from slits 1 and 2 at P be given by

$$E_1 = E_0 \sin \omega t \tag{14.3.6}$$

and

$$E_2 = E_0 \sin(\omega t + \phi) \tag{14.3.7}$$

respectively, where the waves from both slits are assumed have the same amplitude E_0. For simplicity, we have chosen the point P to be the origin, so that the kx dependence in the wave function is eliminated. Since the wave from slit 2 has traveled an extra distance δ to P, E_2 has an extra phase shift ϕ relative to E_1 from slit 1.

For constructive interference, a path difference of $\delta = \lambda$ would correspond to a phase shift of $\phi = 2\pi$. This then implies

$$\frac{\delta}{\lambda} = \frac{\phi}{2\pi} \tag{14.3.8}$$

or

$$\phi = \frac{2\pi}{\lambda}\delta = \frac{2\pi}{\lambda} d\sin\theta \tag{14.3.9}$$

Assuming that both fields point in the same direction, the total electric field may be obtained by using the superposition principle discussed in Section 13.4.1:

$$E = E_1 + E_2 = E_0\left[\sin\omega t + \sin(\omega t + \phi)\right] = 2E_0\cos\left(\frac{\phi}{2}\right)\sin\left(\omega t + \frac{\phi}{2}\right) \tag{14.3.10}$$

where we have used the trigonometric identity

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{14.3.11}$$

The intensity I is proportional to the time average of the square of the total electric field:

$$I \propto \left\langle E^2 \right\rangle = 4E_0^{\,2}\cos^2\left(\frac{\phi}{2}\right)\left\langle \sin^2\left(\omega t + \frac{\phi}{2}\right)\right\rangle = 2E_0^{\,2}\cos^2\left(\frac{\phi}{2}\right) \tag{14.3.12}$$

or

$$I = I_0\cos^2\left(\frac{\phi}{2}\right) \tag{14.3.13}$$

where I_0 is the maximum intensity on the screen. Upon substituting Eq. (14.3.4), the above expression becomes

$$I = I_0\cos^2\left(\frac{\pi d\sin\theta}{\lambda}\right) \tag{14.3.14}$$

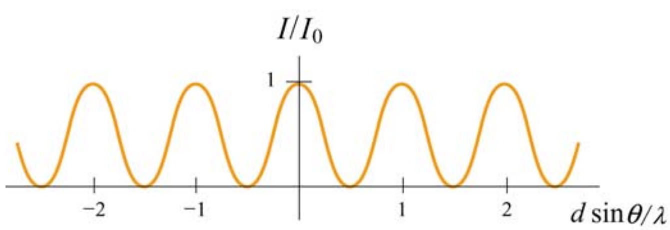


Figure 14.3.2 Intensity as a function of $d\sin\theta/\lambda$

For small angle θ, using Eq. (14.2.5) the intensity can be rewritten as

$$I = I_0\cos^2\left(\frac{\pi d}{\lambda L}y\right) \tag{14.3.15}$$

Example 14.2: Intensity of Three-Slit Interference

Suppose a monochromatic coherent source of light passes through three parallel slits, each separated by a distance *d* from its neighbor, as shown in Figure 14.3.3.

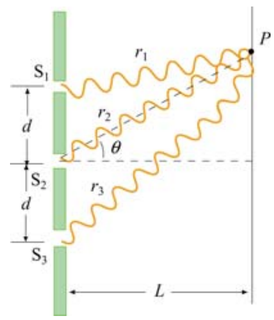


Figure 14.3.3 Three-slit interference.

The waves have the same amplitude E_0 and angular frequency ω, but a constant phase difference $\phi = 2\pi d \sin\theta / \lambda$.

(a) Show that the intensity is

$$I = \frac{I_0}{9}\left[1 + 2\cos\left(\frac{2\pi d \sin\theta}{\lambda}\right)\right]^2 \qquad (14.3.16)$$

where I_0 is the maximum intensity associated with the primary maxima.

(b) What is the ratio of the intensities of the primary and secondary maxima?

Solutions:

(a) Let the three waves emerging from the slits be

$$E_1 = E_0 \sin \omega t, \;\; E_2 = E_0 \sin(\omega t + \phi), \;\; E_3 = E_0 \sin(\omega t + 2\phi) \qquad (14.3.17)$$

Using the trigonometric identity

$$\sin\alpha + \sin\beta = 2\cos\left(\frac{\alpha - \beta}{2}\right)\sin\left(\frac{\alpha + \beta}{2}\right) \qquad (14.3.18)$$

the sum of E_1 and E_3 is

$$E_1 + E_3 = E_0\left[\sin\omega t + \sin(\omega t + 2\phi)\right] = 2E_0 \cos\phi \sin(\omega t + \phi) \qquad (14.3.19)$$

The total electric field at the point *P* on the screen is

$$E = E_1 + E_2 + E_3 = 2E_0 \cos\phi \sin(\omega t + \phi) + E_0 \sin(\omega t + \phi)$$
$$= E_0 (1 + 2\cos\phi)\sin(\omega t + \phi) \tag{14.3.20}$$

where $\phi = 2\pi d \sin\theta / \lambda$. The intensity is proportional to $\langle E^2 \rangle$:

$$I \propto E_0^2 (1 + 2\cos\phi)^2 \langle \sin^2(\omega t + \phi) \rangle = \frac{E_0^2}{2}(1 + 2\cos\phi)^2 \tag{14.3.21}$$

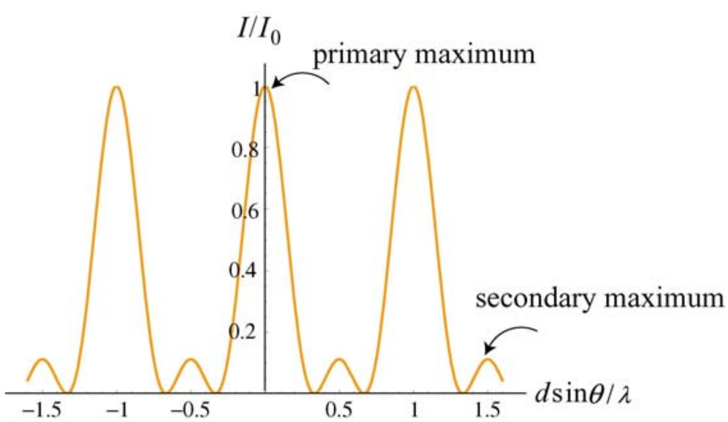
where we have used $\langle \sin^2(\omega t + \phi) \rangle = 1/2$. The maximum intensity I_0 is attained when $\cos\phi = 1$. Thus,

$$\frac{I}{I_0} = \frac{(1 + 2\cos\phi)^2}{9} \tag{14.3.22}$$

which implies

$$I = \frac{I_0}{9}(1 + 2\cos\phi)^2 = \frac{I_0}{9}\left[1 + 2\cos\left(\frac{2\pi d \sin\theta}{\lambda}\right)\right]^2 \tag{14.3.23}$$

(b) The interference pattern is shown in Figure 14.3.4.



From the figure, we see that the minimum intensity is zero, and occurs when $\cos\phi = -1/2$. The condition for primary maxima is $\cos\phi = +1$, which gives $I/I_0 = 1$. In addition, there are also secondary maxima which are located at $\cos\phi = -1$. The condition implies $\phi = (2m+1)\pi$, or $d\sin\theta/\lambda = (m + 1/2)$, $m = 0, \pm 1, \pm 2, ...$ The intensity ratio is $I/I_0 = 1/9$.

14.4 Diffraction

In addition to interference, waves also exhibit another property – *diffraction*, which is the bending of waves as they pass by some objects or through an aperture. The phenomenon of diffraction can be understood using *Huygens's principle* which states that

Every unobstructed point on a wavefront will act a source of secondary spherical waves. The new wavefront is the surface tangent to all the secondary spherical waves.

Figure 14.4.1 illustrates the propagation of the wave based on Huygens's principle.

Figure 14.4.1 Propagation of wave based on Huygens's principle.

According to Huygens's principle, light waves incident on two slits will spread out and exhibit an interference pattern in the region beyond (Figure 14.4.2a). The pattern is called a diffraction pattern. On the other hand, if no bending occurs and the light wave continue to travel in straight lines, then no diffraction pattern would be observed (Figure 14.4.2b).

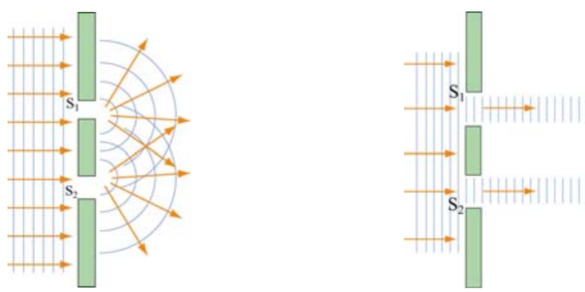


Figure 14.4.2 (a) Spreading of light leading to a diffraction pattern. (b) Absence of diffraction pattern if the paths of the light wave are straight lines.

We shall restrict ourselves to a special case of diffraction called the *Fraunhofer diffraction.* In this case, all light rays that emerge from the slit are approximately parallel to each other. For a diffraction pattern to appear on the screen, a convex lens is placed between the slit and screen to provide convergence of the light rays.

14.5 Single-Slit Diffraction

In our consideration of the Young's double-slit experiments, we have assumed the width of the slits to be so small that each slit is a point source. In this section we shall take the width of slit to be finite and see how Fraunhofer diffraction arises.

Let a source of monochromatic light be incident on a slit of finite width *a*, as shown in Figure 14.5.1.

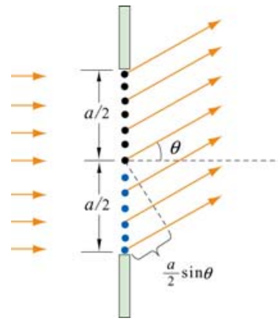


Figure 14.5.1 Diffraction of light by a slit of width *a*.

In diffraction of Fraunhofer type, all rays passing through the slit are approximately parallel. In addition, each portion of the slit will act as a source of light waves according to Huygens's principle. For simplicity we divide the slit into two halves. At the first minimum, each ray from the upper half will be exactly 180° out of phase with a corresponding ray form the lower half. For example, suppose there are 100 point sources, with the first 50 in the lower half, and 51 to 100 in the upper half. Source 1 and source 51 are separated by a distance $a/2$ and are out of phase with a path difference $\delta = \lambda/2$. Similar observation applies to source 2 and source 52, as well as any pair that are a distance $a/2$ apart. Thus, the condition for the first minimum is

$$\frac{a}{2}\sin\theta = \frac{\lambda}{2} \tag{14.5.1}$$

or

$$\sin\theta = \frac{\lambda}{a} \tag{14.5.2}$$

Applying the same reasoning to the wavefronts from four equally spaced points a distance $a/4$ apart, the path difference would be $\delta = a\sin\theta/4$, and the condition for destructive interference is

$$\sin\theta = \frac{2\lambda}{a} \tag{14.5.3}$$

The argument can be generalized to show that destructive interference will occur when

$$\boxed{a\sin\theta = m\lambda, \quad m = \pm 1, \pm 2, \pm 3, \ldots \text{ (destructive interference)}} \tag{14.5.4}$$

Figure 14.5.2 illustrates the intensity distribution for a single-slit diffraction. Note that $\theta = 0$ is a maximum.

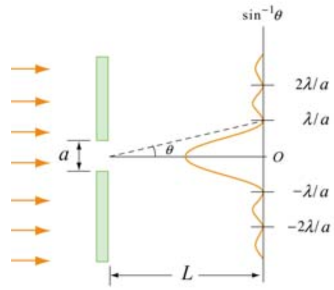


Figure 14.5.2 Intensity distribution for a single-slit diffraction.

By comparing Eq. (14.5.4) with Eq. (14.2.5), we see that the condition for *minima* of a single-slit diffraction becomes the condition for *maxima* of a double-slit interference when the width of a single slit *a* is replaced by the separation between the two slits *d*. The reason is that in the double-slit case, the slits are taken to be so small that each one is considered as a single light source, and the interference of waves originating within the same slit can be neglected. On the other hand, the minimum condition for the single-slit diffraction is obtained precisely by taking into consideration the interference of waves that originate within the *same* slit.

Example 14.3: Single-Slit Diffraction

A monochromatic light with a wavelength of λ = 600 nm passes through a single slit which has a width of 0.800 mm.

(a) What is the distance between the slit and the screen be located if the first minimum in the diffraction pattern is at a distance 1.00 mm from the center of the screen?

(b) Calculate the width of the central maximum.

Solutions:

(a) The general condition for destructive interference is

$$\sin\theta = m\frac{\lambda}{a} \qquad m = \pm 1,\ \pm 2,\ \pm 3,\ ...$$

For small θ, we employ the approximation $\sin\theta \approx \tan\theta = y/L$, which yields

$$\frac{y}{L} \approx m\frac{\lambda}{a}$$

The first minimum corresponds to $m=1$. If $y_1 = 1.00\text{ mm}$, then

$$L = \frac{a y_1}{m\lambda} = \frac{\left(8.00\times10^{-4}\text{ m}\right)\left(1.00\times10^{-3}\text{ m}\right)}{1\left(600\times10^{-9}\text{ m}\right)} = 1.33\text{ m}$$

(b) The width of the central maximum is (see Figure 14.5.2)

$$w = 2y_1 = 2\left(1.00\times10^{-3}\text{ m}\right) = 2.00\text{ mm}$$

14.6 Intensity of Single-Slit Diffraction

How do we determine the intensity distribution for the pattern produced by a single-slit diffraction? To calculate this, we must find the total electric field by adding the field contributions from each point.

Let's divide the single slit into N small zones each of width $\Delta y = a/N$, as shown in Figure 14.6.1. The convex lens is used to bring parallel light rays to a focal point P on the screen. We shall assume that $\Delta y \ll \lambda$ so that all the light from a given zone is in phase. Two adjacent zones have a relative path length $\delta = \Delta y \sin\theta$. The relative phase shift $\Delta\beta$ is given by the ratio

$$\frac{\Delta\beta}{2\pi} = \frac{\delta}{\lambda} = \frac{\Delta y \sin\theta}{\lambda}, \quad \Rightarrow \quad \Delta\beta = \frac{2\pi}{\lambda}\Delta y \sin\theta \tag{14.6.1}$$

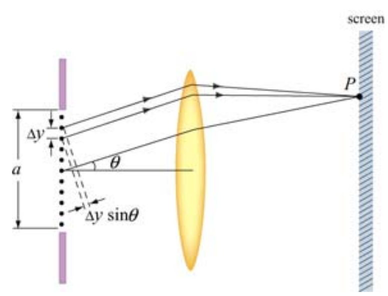


Figure 14.6.1 Single-slit Fraunhofer diffraction

Suppose the wavefront from the first point (counting from the top) arrives at the point P on the screen with an electric field given by

$$E_1 = E_{10}\sin\omega t \tag{14.6.2}$$

The electric field from point 2 adjacent to point 1 will have a phase shift $\Delta\beta$, and the field is

$$E_2 = E_{10}\sin\left(\omega t + \Delta\beta\right) \tag{14.6.3}$$

Since each successive component has the same phase shift relative the previous one, the electric field from point N is

$$E_N = E_{10}\sin\left(\omega t + (N-1)\Delta\beta\right) \tag{14.6.4}$$

The total electric field is the sum of each individual contribution:

$$E = E_1 + E_2 + \cdots E_N = E_{10}\left[\sin\omega t + \sin\left(\omega t + \Delta\beta\right) + \cdots + \sin\left(\omega t + (N-1)\Delta\beta\right)\right] \tag{14.6.5}$$

Note that total phase shift between the point N and the point 1 is

$$\beta = N\Delta\beta = \frac{2\pi}{\lambda}N\Delta y\sin\theta = \frac{2\pi}{\lambda}a\sin\theta \tag{14.6.6}$$

where $N\Delta y = a$. The expression for the total field given in Eq. (14.6.5) can be simplified using some algebra and the trigonometric relation

$$\cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin\alpha\sin\beta \tag{14.6.7}$$

[See Appendix for alternative approaches to simplifying Eq. (14.6.5).] To use the above in Eq. (14.6.5), consider

$$\begin{aligned}
\cos(\omega t - \Delta\beta/2) - \cos(\omega t + \Delta\beta/2) &= 2\sin\omega t\sin(\Delta\beta/2)\\
\cos(\omega t + \Delta\beta/2) - \cos(\omega t + 3\Delta\beta/2) &= 2\sin(\omega t + \Delta\beta)\sin(\Delta\beta/2)\\
\cos(\omega t + 3\Delta\beta/2) - \cos(\omega t + 5\Delta\beta/2) &= 2\sin(\omega t + 2\Delta\beta)\sin(\Delta\beta/2)\\
&\vdots\\
\cos[\omega t + (N-1/2)\Delta\beta] - \cos[\omega t + (N-3/2)\Delta\beta] &= 2\sin[\omega t + (N-1)\Delta\beta]\sin(\Delta\beta/2)
\end{aligned} \tag{14.6.8}$$

Adding the terms and noting that all but two terms on the left cancel leads to

$$\begin{aligned}
&\cos(\omega t - \Delta\beta/2) - \cos[\omega t - (N-1/2)\Delta\beta]\\
&= 2\sin(\Delta\beta/2)\left[\sin\omega t + \sin\left(\omega t + \Delta\beta\right) + \cdots + \sin\left(\omega t + (N-1)\Delta\beta\right)\right]
\end{aligned} \tag{14.6.9}$$

The two terms on the left combine to

$$\begin{aligned}
&\cos(\omega t - \Delta\beta/2) - \cos[\omega t - (N-1/2)\Delta\beta]\\
&= 2\sin(\omega t + (N-1)\Delta\beta/2)\sin(N\Delta\beta/2)
\end{aligned} \tag{14.6.10}$$

with the result that

$$\left[\sin\omega t+\sin\left(\omega t+\Delta\beta\right)+\cdots+\sin\left(\omega t+(N-1)\Delta\beta\right)\right]$$
$$=\frac{\sin[\omega t+(N-1)\Delta\beta/2]\sin(\beta/2)}{\sin(\Delta\beta/2)} \tag{14.6.11}$$

The total electric field then becomes

$$E=E_{10}\left[\frac{\sin(\beta/2)}{\sin(\Delta\beta/2)}\right]\sin\left(\omega t+(N-1)\Delta\beta/2\right) \tag{14.6.12}$$

The intensity *I* is proportional to the time average of E^2:

$$\left\langle E^2\right\rangle=E_{10}^2\left[\frac{\sin(\beta/2)}{\sin(\Delta\beta/2)}\right]^2\left\langle\sin^2\left(\omega t+(N-1)\Delta\beta/2\right)\right\rangle=\frac{1}{2}E_{10}^2\left[\frac{\sin(\beta/2)}{\sin(\Delta\beta/2)}\right]^2 \tag{14.6.13}$$

and we express I as

$$I=\frac{I_0}{N^2}\left[\frac{\sin(\beta/2)}{\sin(\Delta\beta/2)}\right]^2 \tag{14.6.14}$$

where the extra factor N^2 has been inserted to ensure that I_0 corresponds to the intensity at the central maximum $\beta=0$ $(\theta=0)$. In the limit where $\Delta\beta\to 0$,

$$N\sin(\Delta\beta/2)\approx N\Delta\beta/2=\beta/2 \tag{14.6.15}$$

and the intensity becomes

$$I=I_0\left[\frac{\sin\left(\beta/2\right)}{\beta/2}\right]^2=I_0\left[\frac{\sin\left(\pi a\sin\theta/\lambda\right)}{\pi a\sin\theta/\lambda}\right]^2 \tag{14.6.16}$$

In Figure 14.6.2, we plot the ratio of the intensity I/I_0 as a function of $\beta/2$.

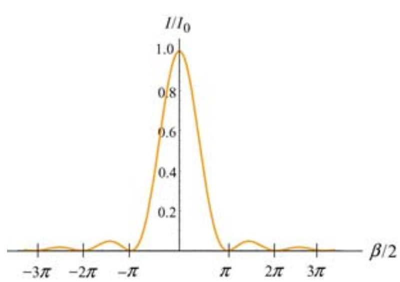


Figure 14.6.2 Intensity of the single-slit Fraunhofer diffraction pattern.

From Eq. (14.6.15), we readily see that the condition for minimum intensity is

$$\frac{\pi}{\lambda} a \sin\theta = m\pi, \quad m = \pm 1, \pm 2, \pm 3, \ldots .$$

or

$$\boxed{\sin\theta = m\frac{\lambda}{a}, \quad m = \pm 1, \pm 2, \pm 3, \ldots} \qquad (14.6.17)$$

In Figure 14.6.3 the intensity is plotted as a function of the angle θ, for $a = \lambda$ and $a = 2\lambda$. We see that as the ratio a/λ grows, the peak becomes narrower, and more light is concentrated in the central peak. In this case, the variation of I_0 with the width a is not shown.

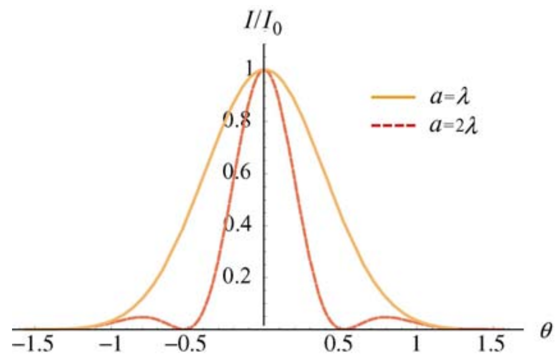


Figure 14.6.3 Intensity of single-slit diffraction as a function of θ for $a = \lambda$ and $a = 2\lambda$.

14.7 Intensity of Double-Slit Diffraction Patterns

In the previous sections, we have seen that the intensities of the single-slit diffraction and the double-slit interference are given by:

$$I = I_0 \left[\frac{\sin(\pi a \sin\theta/\lambda)}{\pi a \sin\theta/\lambda}\right]^2 \qquad \text{single-slit diffraction}$$

$$I = I_0 \cos^2\left(\frac{\phi}{2}\right) = I_0 \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right) \qquad \text{double-slit interference}$$

Suppose we now have two slits, each having a width *a*, and separated by a distance d. The resulting interference pattern for the double-slit will also include a diffraction pattern due to the individual slit. The intensity of the total pattern is simply the product of the two functions:

$$I = I_0 \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right)\left[\frac{\sin(\pi a \sin\theta/\lambda)}{\pi a \sin\theta/\lambda}\right]^2 \quad (14.7.1)$$

The first and the second terms in the above equation are referred to as the "interference factor" and the "diffraction factor," respectively. While the former yields the interference substructure, the latter acts as an envelope which sets limits on the number of the interference peaks (see Figure 14.7.1).

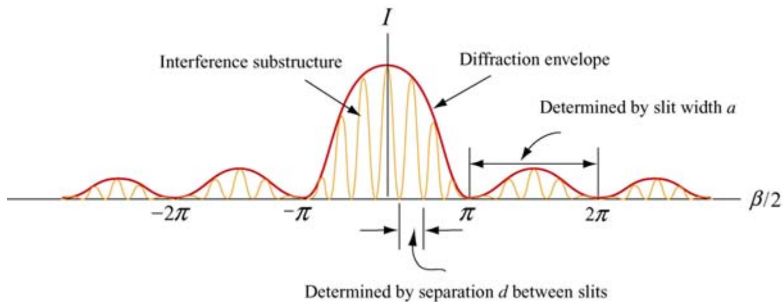


Figure 14.7.1 Double-slit interference with diffraction.

We have seen that the interference maxima occur when $d\sin\theta = m\lambda$. On the other hand, the condition for the first diffraction minimum is $a\sin\theta = \lambda$. Thus, a particular interference maximum with order number *m* may coincide with the first diffraction minimum. The value of *m* may be obtained as:

$$\frac{d\sin\theta}{a\sin\theta} = \frac{m\lambda}{\lambda}$$

or

$$\boxed{m = \frac{d}{a}} \quad (14.7.2)$$

Since the *m*th fringe is not seen, the number of fringes on each side of the central fringe is $m-1$. Thus, the total number of fringes in the central diffraction maximum is

$$N = 2(m+1)+1 = 2m-1 \quad (14.7.3)$$

14.8 Diffraction Grating

A diffraction grating consists of a large number N of slits each of width a and separated from the next by a distance d, as shown in Figure 14.8.1.

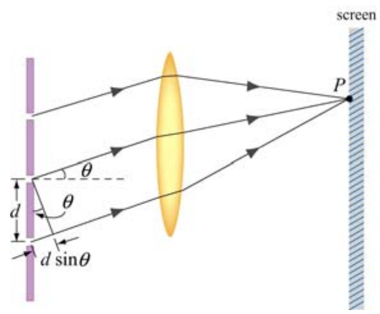


Figure 14.8.1 Diffraction grating

If we assume that the incident light is planar and diffraction spreads the light from each slit over a wide angle so that the light from all the slits will interfere with each other. The relative path difference between each pair of adjacent slits is $\delta = d\sin\theta$, similar to the calculation we made for the double-slit case. If this path difference is equal to an integral multiple of wavelengths then all the slits will constructively interfere with each other and a bright spot will appear on the screen at an angle θ. Thus, the condition for the principal maxima is given by

$$d\sin\theta = m\lambda, \quad m = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{14.8.1}$$

If the wavelength of the light and the location of the *m*-order maximum are known, the distance d between slits may be readily deduced.

The location of the maxima does not depend on the number of slits, *N*. However, the maxima become sharper and more intense as *N* is increased. The width of the maxima can be shown to be inversely proportional to *N*. In Figure 14.8.2, we show the intensity distribution as a function of $\beta/2$ for diffraction grating with $N = 10$ and $N = 30$. Notice that the principal maxima become sharper and narrower as N increases.

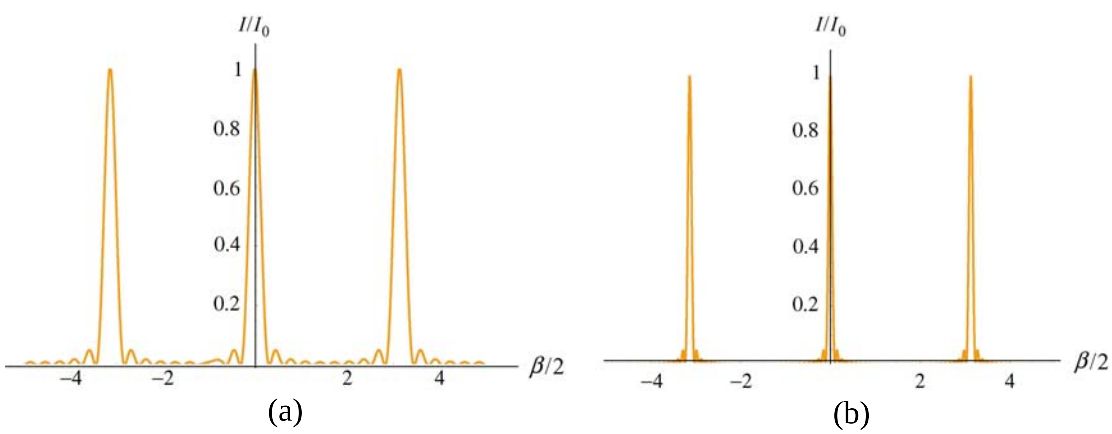


Figure 14.8.2 Intensity distribution for a diffraction grating for (a) $N = 10$ and (b) $N = 30$.

The observation can be explained as follows: suppose an angle θ (recall that $\beta = 2\pi a \sin\theta / \lambda$) which initially gives a principal maximum is increased slightly, if there were only two slits, then the two waves will still be nearly in phase and produce maxima which are broad. However, in grating with a large number of slits, even though θ may only be slightly deviated from the value that produces a maximum, it could be exactly out of phase with light wave from another slit far away. Since grating produces peaks that are much sharper than the two-slit system, it gives a more precise measurement of the wavelength.

14.9 Summary

- **Interference** is the combination of two or more waves to form a composite wave based on the superposition principle.

- In **Young's double-slit experiment**, where a coherent monochromatic light source with wavelength λ emerges from two slits that are separated by a distance *d*, the condition for **constructive interference** is

  $$\delta = d\sin\theta = m\lambda, \quad m = 0, \pm 1, \pm 2, \pm 3, \ldots \text{ (constructive interference)}$$

  where *m* is called the **order number**. On the other hand, the condition for **destructive interference** is

  $$d\sin\theta = \left(m + \frac{1}{2}\right)\lambda, \quad m = 0, \pm 1, \pm 2, \pm 3, \ldots \text{ (destructive interference)}$$

- The **intensity** in the double-slit interference pattern is

  $$I = I_0 \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right)$$

  where I_0 is the maximum intensity on the screen.

- **Diffraction** is the bending of waves as they pass by an object or through an aperture. In a single-slit Fraunhofer diffraction, the condition for destructive interference is

  $$\sin\theta = m\frac{\lambda}{a}, \quad m = \pm 1, \pm 2, \pm 3, \ldots \text{ (destructive interference)}$$

  where a is the width of the slit. The intensity of the interference pattern is

$$I = I_0\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2 = I_0\left[\frac{\sin(\pi a\sin\theta/\lambda)}{\pi a\sin\theta/\lambda}\right]^2$$

where $\beta = 2\pi a\sin\theta/\lambda$ is the total phase difference between waves from the upper end and the lower end of the slit, and I_0 is the intensity at $\theta = 0$.

- For two slits each having a width a and separated by a distance d, the interference pattern will also include a diffraction pattern due to the single slit, and the intensity is

$$I = I_0\cos^2\left(\frac{\pi d\sin\theta}{\lambda}\right)\left[\frac{\sin(\pi a\sin\theta/\lambda)}{\pi a\sin\theta/\lambda}\right]^2$$

14.10 Appendix: Computing the Total Electric Field

Section 14.6 we used a trigonometric relation and obtained the total electric field for a single-slit diffraction. Below we show two alternative approaches of how Eq. (14.6.5) can be simplified.

(1) Complex representation:

The total field E may be regarded as a geometric series. From the Euler formula

$$e^{ix} = \sum_{n=0}^{\infty}\frac{(ix)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} + i\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \cos x + i\sin x \qquad (14.10.1)$$

we may write

$$\sin x = \text{Im}(e^{ix}) \qquad (14.10.2)$$

where the notation "Im" stands for the imaginary part. Thus, we have

$$\begin{aligned}
&\sin\omega t + \sin(\omega t + \Delta\beta) + \ldots + \sin(\omega t + (N-1)\Delta\beta) \\
&= \text{Im}\left[e^{i\omega t} + e^{i(\omega t+\Delta\beta)} + \ldots + e^{i(\omega t+(N-1)\Delta\beta)}\right] = \text{Im}\left[e^{i\omega t}(1 + e^{i\Delta\beta} + \ldots + e^{i(N-1)\Delta\beta})\right] \\
&= \text{Im}\left[e^{i\omega t}\frac{1-e^{iN\Delta\beta}}{1-e^{i\Delta\beta}}\right] = \text{Im}\left[e^{i\omega t}\frac{-e^{iN\Delta\beta/2}(e^{iN\Delta\beta/2} - e^{-iN\Delta\beta/2})}{-e^{i\Delta\beta/2}(e^{i\Delta\beta/2} - e^{-i\Delta\beta/2})}\right] \\
&= \text{Im}\left[e^{i(\omega t+(N-1)\Delta\beta/2)}\frac{\sin(\beta/2)}{\sin(\Delta\beta/2)}\right] = \sin(\omega t + (N-1)\Delta\beta/2)\frac{\sin(\beta/2)}{\sin(\Delta\beta/2)}
\end{aligned} \qquad (14.10.3)$$

where we have used

$$\sum_{n=0}^{N} a^n = 1 + a + a^2 + \ldots = \frac{1-a^{n+1}}{1-a}, \quad |a| < 1 \tag{14.10.4}$$

The total electric field then becomes

$$E = E_{10}\left[\frac{\sin(\beta/2)}{\sin(\Delta\beta/2)}\right]\sin\left(\omega t + (N-1)\Delta\beta/2\right) \tag{14.10.5}$$

which is the same as that given in Eq. (14.6.12).

(2) Phasor diagram:

Alternatively, we may also use phasor diagrams to obtain the time-independent portion of the resultant field. Before doing this, let's first see how phasor addition works for two wave functions.

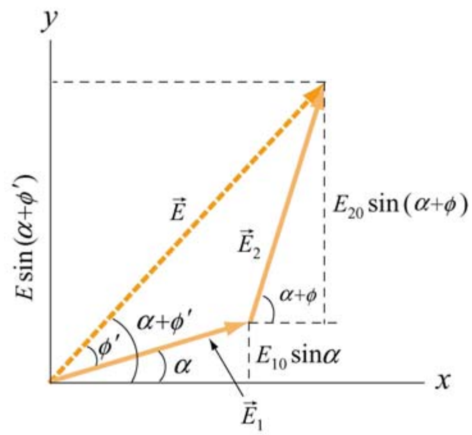


Figure 14.10.1 Addition of two phasors.

Let $E_1 = E_{10}\sin\alpha$ and $E_2 = E_{20}\sin(\alpha+\phi)$, with the total field given by

$$E = E_1 + E_2 = E_{10}\sin\alpha + E_{20}\sin(\alpha+\phi) = E_0\sin(\alpha+\phi') \tag{14.10.6}$$

Using the phasor approach, the fields E_1 and E_2 are represented by two-dimensional vectors $\vec{E}_1$ and $\vec{E}_2$, respectively. The addition of $\vec{E} = \vec{E}_1 + \vec{E}_2$ is shown in Figure 14.10.1.

The idea of this geometric approach is based on the fact that when adding two vectors, the component of the resultant vector is equal to the sum of the individual components. The vertical component of $\vec{E}$, which is the sum of the vertical projections of $\vec{E}_1$ and $\vec{E}_2$, is the resultant field E.

If the two fields have the same amplitude $E_{10} = E_{20}$, the phasor diagram becomes

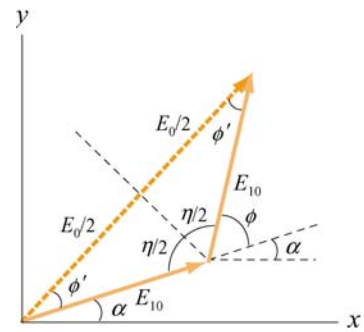


Figure 14.10.2 Addition of two phasors with the same amplitude.

From the diagram, we see that $\eta + \phi = \pi$ and $2\phi' + \eta = \pi$. This gives

$$\phi' = \frac{\pi}{2} - \frac{\eta}{2} = \frac{\pi}{2} - \frac{1}{2}(\pi - \phi) = \frac{\phi}{2} \tag{14.10.7}$$

In addition,

$$\cos\phi' = \frac{E_0/2}{E_{10}} \tag{14.10.8}$$

Combining the two equations, we obtain

$$E_0 = 2E_{10}\cos\phi' = 2E_{10}\cos\left(\frac{\phi}{2}\right) \tag{14.10.9}$$

and the resultant field is

$$E = 2E_{10}\cos\left(\frac{\phi}{2}\right)\sin\left(\alpha + \frac{\phi}{2}\right) \tag{14.10.10}$$

One may also obtain the total field using the trigonometric identity given in Eq. (14.3.18).

Now let's turn to the situation where there are N sources, as in our calculation of the intensity of the single-slit diffraction intensity in Section 14.6. By setting $t = 0$ in Eq. (14.6.5), the time-independent part of the total field is

$$E = E_1 + E_2 + \cdots E_N = E_{10}\left[\sin(\Delta\beta) + \cdots + \sin((N-1)\Delta\beta)\right] \tag{14.10.11}$$

The corresponding phasor diagram is shown in Figure 14.10.3. Notice that all the phasors lie on a circular arc of radius R, with each successive phasor differed in phase by $\Delta\beta$.

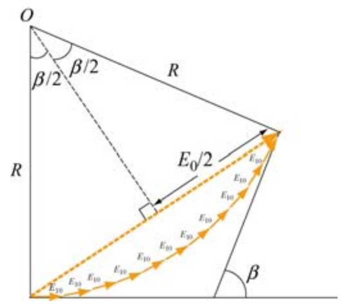


Figure 14.10.3 Phasor diagram for determining the time-independent portion of E .

From the figure, we see that

$$\sin\frac{\beta}{2}=\frac{E_0/2}{R} \tag{14.10.12}$$

Since the length of the arc is $NE_{10}=R\beta$, we have

$$E_0=2R\sin\left(\frac{\beta}{2}\right)=2\frac{NE_{10}}{\beta}\sin\left(\frac{\beta}{2}\right)=E_{10}\left[\frac{\sin(\beta/2)}{\Delta\beta/2}\right] \tag{14.10.13}$$

where $\beta=N\Delta\beta$. The result is completely consistent with that obtained in Eq. (14.6.11) using the algebraic approach. The intensity is proportional to E_0^2 , and again we write it as

$$I=\frac{I_0}{N^2}\left[\frac{\sin(\beta/2)}{\Delta\beta/2}\right]^2=I_0\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2 \tag{14.10.14}$$

which reproduces the result shown in Eq. (14.6.15).

14.11 Solved Problems

14.11.1 Double-Slit Experiment

In Young's double-slit experiment, suppose the separation between the two slits is d=0.320 mm. If a beam of 500-nm light strikes the slits and produces an interference pattern. How many maxima will there be in the angular range $-45.0°<\theta<45.0°$?

Solution:

On the viewing screen, light intensity is a maximum when the two waves interfere constructively. This occurs when

$$d\sin\theta = m\lambda, \quad m=0,\pm 1,\pm 2,... \tag{14.11.1}$$

where λ is the wavelength of the light. At $\theta = 45.0°$, $d = 3.20\times 10^{-4}$ m and $\lambda = 500\times 10^{-9}$ m, we get

$$m = \frac{d\sin\theta}{\lambda} = 452.5 \tag{14.11.2}$$

Thus, there are 452 maxima in the range $0 < \theta < 45.0°$. By symmetry, there are also 452 maxima in the range $-45.0° < \theta < 0$. Including the one for $m = 0$ straight ahead, the total number of maxima is

$$N = 452 + 452 + 1 = 905 \tag{14.11.3}$$

14.11.2 Phase Difference

In the double-slit interference experiment shown in Figure 14.2.3, suppose $d = 0.100$ mm and $L = 1.00$ m, and the incident light is monochromatic with a wavelength λ=500 nm.

(a) What is the phase difference between the two waves arriving at a point P on the screen when $\theta = 0.800°$?

(b) What is the phase difference between the two waves arriving at a point P on the screen when $y = 4.00$ mm?

(c) If ϕ=1/3 rad, what is the value of θ?

(d) If the path difference is $\delta = \lambda/4$, what is the value of θ?

Solutions:

(a) The phase difference ϕ between the two wavefronts is given by

$$\phi = \frac{2\pi}{\lambda}\delta = \frac{2\pi}{\lambda} d\sin\theta \tag{14.11.4}$$

With $\theta = 0.800°$, we have

$$\phi = \frac{2\pi}{(5.00\times 10^{-7}\text{ m})}(1.00\times 10^{-4}\text{ m})\sin(0.800°) = 17.5 \text{ rad} \tag{14.11.5}$$

(b) When θ is small, we make use of the approximation $\sin\theta \approx \tan\theta = y/L$. Thus, the phase difference becomes

$$\phi \approx \frac{2\pi}{\lambda} d \left(\frac{y}{L} \right) \tag{14.11.6}$$

For $y = 4.00\,\text{mm}$, we have

$$\phi = \frac{2\pi}{(5.00\times10^{-7}\ \text{m})}(1.00\times10^{-4}\ \text{m})\left(\frac{4.00\times10^{-3}\ \text{m}}{1.00\,\text{m}}\right) = 5.03\ \text{rad} \tag{14.11.7}$$

(c) For $\phi = 1/3\ \text{rad}$, we have

$$\frac{1}{3}\ \text{rad} = \frac{2\pi}{\lambda} d\sin\theta = \frac{2\pi}{(5.00\times10^{-7}\ \text{m})}(1.00\times10^{-4}\ \text{m})\sin\theta \tag{14.11.8}$$

which gives

$$\theta = 0.0152^\circ \tag{14.11.9}$$

(d) For $\delta = d\sin\theta = \lambda/4$, we have

$$\theta = \sin^{-1}\left(\frac{\lambda}{4d}\right) = \sin^{-1}\left[\frac{5.00\times10^{-7}\ \text{m}}{4(1.00\times10^{-4}\ \text{m})}\right] = 0.0716^\circ \tag{14.11.10}$$

14.11.3 Constructive Interference

Coherent light rays of wavelength λ are illuminated on a pair of slits separated by distance d at an angle θ_1, as shown in Figure 14.11.1.

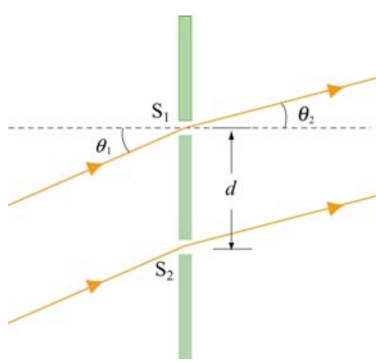


Figure 14.11.1

If an interference maximum is formed at an angle θ_2 at a screen far from the slits, find the relationship between θ_1, θ_2, d and λ.

Solution:

The path difference between the two rays is

$$\delta = d\sin\theta_1 - d\sin\theta_2 \tag{14.11.11}$$

The condition for constructive interference is $\delta = m\lambda$, where $m = 0, \pm 1, \pm 2, \ldots$ is the order number. Thus, we have

$$d\left(\sin\theta_1 - \sin\theta_2\right) = m\lambda \tag{14.11.12}$$

14.11.4 Intensity in Double-Slit Interference

Let the intensity on the screen at a point P in a double-slit interference pattern be 60.0% of the maximum value.

(a) What is the minimum phase difference (in radians) between sources?

(b) In (a), what is the corresponding path difference if the wavelength of the light is $\lambda = 500 \text{ nm}$?

Solution:

(a) The average intensity is given by

$$I = I_0 \cos^2\left(\frac{\phi}{2}\right) \tag{14.11.13}$$

where I_0 is the maximum light intensity. Thus,

$$0.60 = \cos^2\left(\frac{\phi}{2}\right) \tag{14.11.14}$$

which yields

$$\phi = 2\cos^{-1}\left(\sqrt{\frac{I}{I_0}}\right) = 2\cos^{-1}(\sqrt{0.60}) = 78.5° = 1.37 \text{ rad} \tag{14.11.15}$$

(b) The phase difference ϕ is related to the path difference δ and the wavelength λ by

$$\delta = \frac{\lambda\phi}{2\pi} = \frac{(500\,\text{nm})(1.37\,\text{rad})}{2\pi} = 109 \text{ nm} \tag{14.11.16}$$

14.11.5 Second-Order Bright Fringe

A monochromatic light is incident on a single slit of width 0.800 mm, and a diffraction pattern is formed at a screen which is 0.800 m away from the slit. The second-order bright fringe is at a distance 1.60 mm from the center of the central maximum. What is the wavelength of the incident light?

Solution:

The general condition for destructive interference is

$$\sin\theta = m\frac{\lambda}{a} \approx \frac{y}{L} \tag{14.11.17}$$

where small-angle approximation has been made. Thus, the position of the *m*-th order dark fringe measured from the central axis is

$$y_m = m\frac{\lambda L}{a} \tag{14.11.18}$$

Let the second bright fringe be located halfway between the second and the third dark fringes. That is,

$$y_{2b} = \frac{1}{2}(y_2 + y_3) = \frac{1}{2}(2+3)\frac{\lambda L}{a} = \frac{5\lambda L}{2a} \tag{14.11.19}$$

The approximate wavelength of the incident light is then

$$\lambda \approx \frac{2a\,y_{2b}}{5L} = \frac{2(0.800\times10^{-3}\ \text{m})(1.60\times10^{-3}\ \text{m})}{5(0.800\ \text{m})} = 6.40\times10^{-7}\ \text{m} \tag{14.11.20}$$

14.11.6 Intensity in Double-Slit Diffraction

Coherent light with a wavelength of λ = 500 nm is sent through two parallel slits, each having a width $a = 0.700\ \mu$m. The distance between the centers of the slits is $d = 2.80\ \mu\text{m}$. The screen has a semi-cylindrical shape, with its axis at the midline between the slits.

(a) Find the direction of the interference maxima on the screen. Express your answers in terms of the angle away from the bisector of the line joining the slits.

(b) How many bright fringes appear on the screen?

(c) For each bright fringe, find the intensity, measured relative to the intensity I_0 associated with the central maximum.

Solutions:

(a) The condition for double-slit interference maxima is given by

$$d\sin\theta = m\lambda \qquad m = 0,\ \pm1,\ \pm2,... \tag{14.11.21}$$

which yields

$$\theta_m = \sin^{-1}\left(\frac{m\lambda}{d}\right) \tag{14.11.22}$$

With $\lambda = 5.00\times10^{-7}\,\text{m}$ and $d = 2.80\times10^{-6}\,\text{m}$, the above equation becomes

$$\theta_m = \sin^{-1}\left(m\ \frac{5.00\times10^{-7}\,\text{m}}{2.80\times10^{-6}\,\text{m}}\right) = \sin^{-1}(0.179m) \tag{14.11.23}$$

The solutions are

$$\begin{aligned} &\theta_0 = 0^\circ & &\theta_{\pm1} = \sin^{-1}(\pm0.179) = \pm10.3^\circ \\ &\theta_{\pm2} = \sin^{-1}(\pm0.357) = \pm20.9^\circ & &\theta_{\pm3} = \sin^{-1}(\pm0.536) = \pm32.4^\circ \\ &\theta_{\pm4} = \sin^{-1}(\pm0.714) = \pm45.6^\circ & &\theta_{\pm5} = \sin^{-1}(\pm0.893) = \pm63.2^\circ \\ &\theta_{\pm6} = \sin^{-1}(\pm1.07) = \text{ no solution} & & \end{aligned} \tag{14.11.24}$$

Thus, we see that there are a total of 11 directions of interference maxima.

(b) The general condition for single-slit diffraction minima is $a\sin\theta = m\lambda$, or

$$\theta_m = \sin^{-1}\left(\frac{m\lambda}{a}\right) \qquad m = \pm1,\ \pm2,\ ... \tag{14.11.25}$$

With $\lambda = 5.00\times10^{-7}\,\text{m}$ and $a = 7.00\times10^{-7}\,\text{m}$, the above equation becomes

$$\theta_m = \sin^{-1}\left(m\ \frac{5.00\times10^{-7}\,\text{m}}{7.00\times10^{-7}\,\text{m}}\right) = \sin^{-1}(0.714m) \tag{14.11.26}$$

The solutions are

$$\begin{aligned}\theta_{\pm 1} &= \sin^{-1}(\pm 0.714) = \pm 45.6^\circ \\ \theta_{\pm 2} &= \sin^{-1}(\pm 1.43) = \text{ no solution}\end{aligned} \tag{14.11.27}$$

Since these angles correspond to dark fringes, the total number of bright fringes is $N = 11 - 2 = 9$.

(c) The intensity on the screen is given by

$$I = I_0 \left[\frac{\sin\left(\pi a \sin\theta / \lambda\right)}{\pi a \sin\theta / \lambda} \right]^2 \tag{14.11.28}$$

where I_0 is the intensity at $\theta = 0$.

(i) At $\theta = 0^\circ$, we have the central maximum and $I / I_0 = 1.00$.

(ii) At $\theta = \pm 10.3^\circ$, we have $\pi a \sin\theta / \lambda = \pm \dfrac{\pi(0.700\ \mu\text{m}) \sin 10.3^\circ}{0.500\ \mu\text{m}} = \pm 0.785\text{ rad} = \pm 45.0^\circ$, which gives

$$\frac{I}{I_0} = \left[\pm \frac{\sin 45.0^\circ}{0.785} \right]^2 = 0.811 \tag{14.11.29}$$

(iii) At $\theta = \pm 20.9^\circ$, we have $\pi a \sin\theta / \lambda = \pm 1.57\text{ rad} = \pm 90.0^\circ$, and the intensity ratio is

$$\frac{I}{I_0} = \left[\pm \frac{\sin 90.0^\circ}{1.57} \right]^2 = 0.406 \tag{14.11.30}$$

(iv) At $\theta = \pm 32.4^\circ$, we have $\pi a \sin\theta / \lambda = \pm 2.36\text{ rad} = \pm 135^\circ$, and the intensity ratio is

$$\frac{I}{I_0} = \left[\pm \frac{\sin 135^\circ}{2.36} \right]^2 = 0.0901 \tag{14.11.31}$$

(v) At $\theta = \pm 63.2^\circ$, we have $\pi a \sin\theta / \lambda = \pm 3.93\text{ rad} = \pm 225^\circ$, and the intensity ratio is

$$\frac{I}{I_0} = \left[\pm \frac{\sin 225^\circ}{3.93} \right]^2 = 0.0324 \tag{14.11.32}$$

14.12 Conceptual Questions

1. In Young's double-slit experiment, what happens to the spacing between the fringes if

(a) the slit separation is increased?
(b) the wavelength of the incident light is decreased?
(c) if the distance between the slits and the viewing screen is increased?

2. In Young's double-slit experiment, how would the interference pattern change if white light is used?

3. Explain why the light from the two headlights of a distant car does not produce an interference pattern.

4. What happens to the width of the central maximum in a single-slit diffraction if the slit width is increased?

5. In a single-slit diffraction, what happens to the intensity pattern if the slit width becomes narrower and narrower?

6. In calculating the intensity in double-slit interference, can we simply add the intensities from each of the two slits?

14.13 Additional Problems

14.13.1 Double-Slit Interference

In the double-slit interference experiment, suppose the slits are separated by $d = 1.00\,\text{cm}$ and the viewing screen is located at a distance $L = 1.20\,\text{m}$ from the slits. Let the incident light be monochromatic with a wavelength λ=500 nm.

(a) Calculate the spacing between the adjacent bright fringes.

(b) What is the distance between the third-order fringe and the center line?

14.13.2 Interference-Diffraction Pattern

In the double-slit Fraunhofer interference-diffraction experiment, if the slits of width 0.010 mm are separated by a distance 0.20 mm, and the incident light is monochromatic with a wavelength $\lambda = 600\text{ nm}$. How many bright fringes are there in the central diffraction maximum?

14.13.3 Three-Slit Interference

Suppose a monochromatic coherent light source of wavelength λ passes through three parallel slits, each separated by a distance d from its neighbor.

(a) Show that the positions of the interference minima on a viewing screen a distance $L \gg d$ away is approximately given by

$$y = n\frac{\lambda L}{3d}, \quad n = 1, 2, 4, 5, 7, 8, 10, \ldots$$

where n is not a multiple of 3.

(b) Let $L = 1.20 \text{ m}$, $\lambda = 450 \text{ nm}$ and $d = 0.10 \text{ mm}$. What is the spacing between the successive minima?

14.13.4 Intensity of Double-Slit Interference

In the double-slit interference experiment, suppose the slits are of different size, and the fields at a point P on the viewing screen are

$$E_1 = E_{10} \sin \omega t, \qquad E_2 = E_{20} \sin(\omega t + \phi)$$

Show that the intensity at P is

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos\phi$$

where I_1 and I_2 are the intensities due to the light from each slit.

14.13.5 Secondary Maxima

In a single-slit diffraction pattern, we have shown in 14.6 that the intensity is

$$I = I_0 \left[\frac{\sin(\beta/2)}{\beta/2} \right]^2 = I_0 \left[\frac{\sin(\pi a \sin\theta/\lambda)}{\pi a \sin\theta/\lambda} \right]^2$$

(a) Explain why the condition for the secondary maxima is not given by $\beta/2 = (m + 1/2)\pi, \quad m = 1, 2, 3, \ldots$

(b) By differentiating the expression above for I, show that the condition for secondary maxima is

$$\frac{\beta}{2} = \tan\left(\frac{\beta}{2}\right)$$

(c) Plot the curves $y = \beta/2$ and $y = \tan(\beta/2)$. Using a calculator which has a graphing function, or mathematical software, find the values of β at which the two curves intersect, and hence, the values of β for the first and second secondary maxima. Compare your results with $\beta/2 = (m+1/2)\pi$.

14.13.6 Interference-Diffraction Pattern

If there are 7 fringes in the central diffraction maximum in a double-slit interference pattern, what can you conclude about the slit width and separation?