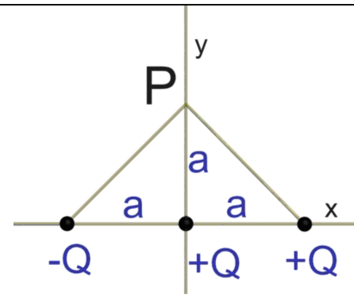
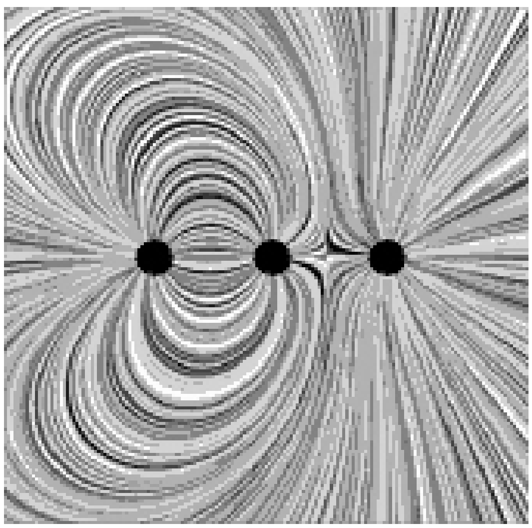
Electric Fields, Dipoles and Torque

Challenge Problem Solutions

Problem 1:

Three charges equal to –Q, +Q and +Q are located a distance a apart along the x axis (see sketch). The point P is located on the positive y-axis a distance a from the origin. (a) What is the electric field $\vec{\mathbf{E}}$ at point P?	

(b) _(f)_ (enter one letter) is the correct field line representation for this problem



Problem 1 Solution:

(a) If we number the charges from left to right, then at point P, the E field due to the charges are

$$\vec{\mathbf{E}}_1 = k_e \frac{-Q}{\left(\sqrt{2}a\right)^3}\left(a\,\hat{\mathbf{i}} + a\,\hat{\mathbf{j}}\right) = -k_e \frac{Q}{2\sqrt{2}a^2}\left(\hat{\mathbf{i}} + \hat{\mathbf{j}}\right)$$

$$\vec{\mathbf{E}}_2 = k_e \frac{Q}{a^2}\left(\hat{\mathbf{j}}\right)$$

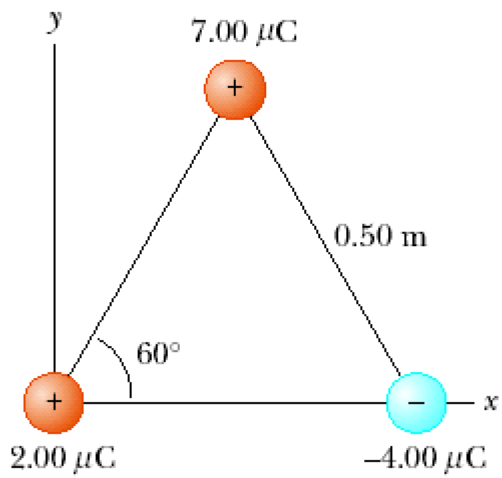
$$\vec{\mathbf{E}}_3 = k_e \frac{Q}{\left(\sqrt{2}a\right)^3}\left(-a\hat{\mathbf{i}} + a\,\hat{\mathbf{j}}\right) = -k_e \frac{Q}{2\sqrt{2}a^2}\left(\hat{\mathbf{i}} - \hat{\mathbf{j}}\right)$$

Thus, the total electric field is $\vec{\mathbf{E}} = \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2 + \vec{\mathbf{E}}_3 = -k_e \frac{Q}{\sqrt{2}a^2}\hat{\mathbf{i}} + k_e \frac{Q}{a^2}\hat{\mathbf{j}}$

_(f)_ is the correct field line representation for this problem
You can tell this because the center and leftmost charges are of opposite signs (field lines start on one and end on the other), while the center and rightmost charges are of the same sign (their field lines repel each other).

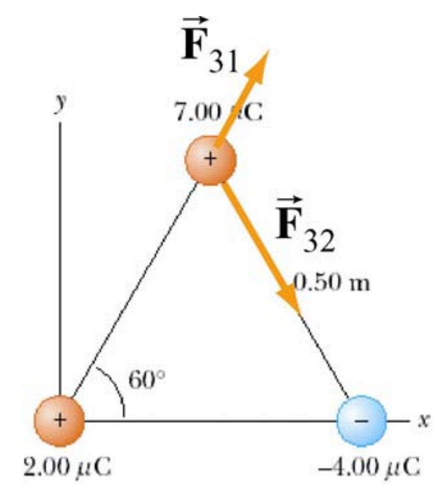
Problem 2:

Consider three point charges located at the corners of an equilateral triangle, as shown in figure. Calculate the resultant electric force on the 7.00-μC charge. Be sure to specify both the magnitude and direction.



Problem 2 Solution:

Consider three point charges located at the corners of an equilateral triangle, as shown in figure below. Calculate the resultant electric force on the 7.00-μC charge. Be sure to specify both the magnitude and direction.



Let $q_1 = 2.00\,\mu\text{C}$, $q_2 = -4.00\,\mu\text{C}$ and $q_3 = 7.00\,\mu\text{C}$. Using Coulomb's law, the magnitudes of the forces exerted on q_3 are

$$F_{31} = \frac{1}{4\pi\varepsilon_0}\frac{q_3 q_1}{r^2} = \frac{\left(8.99\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(7.00\times10^{-6}\ \text{C}\right)\left(2.00\times10^{-6}\ \text{C}\right)}{\left(0.500\ \text{m}\right)^2} = 0.503\ \text{N}$$

$$F_{32} = \frac{1}{4\pi\varepsilon_0}\frac{q_3\,|q_2|}{r^2} = \frac{\left(8.99\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(7.00\times10^{-6}\ \text{C}\right)\left(4.00\times10^{-6}\ \text{C}\right)}{\left(0.500\ \text{m}\right)^2} = 1.01\ \text{N}$$

In Cartesian coordinates, the forces can be written as

$$\vec{\mathbf{F}}_{31} = F_{31}\left(\cos 60°\,\hat{\mathbf{i}} + \sin 60°\,\hat{\mathbf{j}}\right) = \left(0.25\,\text{N}\right)\hat{\mathbf{i}} + \left(0.44\,\text{N}\right)\hat{\mathbf{j}}$$
$$\vec{\mathbf{F}}_{32} = F_{32}\left[\cos(-60°)\,\hat{\mathbf{i}} + \sin(-60°)\,\hat{\mathbf{j}}\right] = \left(0.51\,\text{N}\right)\hat{\mathbf{i}} - \left(0.87\,\text{N}\right)\hat{\mathbf{j}}$$

The total force is

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}_{31} + \vec{\mathbf{F}}_{32} = \left(0.76\ \text{N}\right)\hat{\mathbf{i}} - \left(0.44\ \text{N}\right)\hat{\mathbf{j}}$$

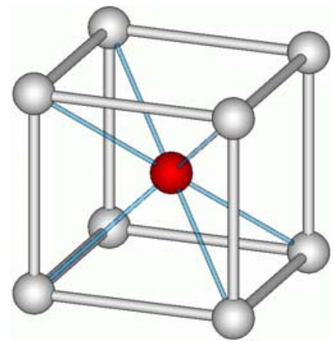
The magnitude of $\vec{\mathbf{F}}$ is

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{(0.76\,\text{N})^2 + (-0.44\,\text{N})^2} = 0.87\,\text{N}$$

and the angle with respect to the +*x* axis is

$$\varphi = \tan^{-1}\left(\frac{F_y}{F_x}\right) = \tan^{-1}\left(\frac{-0.44\,\text{N}}{0.76\,\text{N}}\right) = -30°$$

Problem 3:



Cesium Chloride is a salt with a crystal structure in which cubes of Cs^+ ions (side length a ~ 0.4 nm) surround Cl^- ions, as pictured at right.

(a) What is the magnitude of the net electrostatic force on the Cl^- ion due to its eight nearest neighbor Cs^+ ions?

(b) Occasionally defects arise in which one of the Cs^+ ions is missing. We call this a vacancy. In this case, what is the magnitude and direction (relative to the vacancy) of the net electrostatic force on the Cl^- ion due to its remaining seven

Problem 3 Solution:

(a) By symmetry the electrostatic force from each Cs^+ ion is cancelled by its partner opposite the Cl^- ion, so the net force is $\boxed{\textbf{zero}}$.

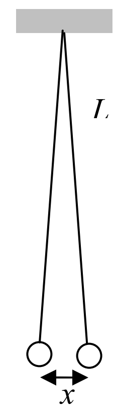
(b) Taking away a Cs^+ ion to create a vacancy is, electronically, the same thing as adding a negative charge to the system at the location of the vacancy (this will cancel out the positive charge of the ion, essentially removing it). This negative charge, $q = -e$, will create a repulsive force on the Cl^- ion, a distance $d = \sqrt{3(a/2)^2}$ away, of magnitude:

$$F = \frac{ke^2}{d^2} = \frac{ke^2}{3(a/2)^2} = \frac{\left(9\times10^9 \text{ Nm}^2\text{C}^{-2}\right)\left(1.6\times10^{-19} \text{ C}\right)^2}{3\left(2\times10^{-10} \text{ m}\right)^2} = \boxed{1.9\times10^{-9} \text{ N}}$$

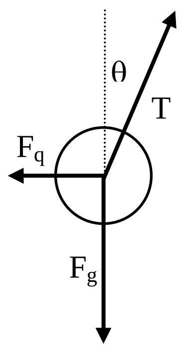
Problem 4:
One version of an electroscope consists of two small conducting balls of mass *m* hanging on **long** strings of length *L*. If a charge $2q$ is transferred to the system (so each ball acquires the same charge *q*), they will repel.

(a) Neglecting gravitational attraction between the balls, by what distance *x* do they move apart when charged?
(b) Is it reasonable to ignore this gravitational attraction? More precisely, if we were to put a very small charge, say one electron, on each of the balls, how light would the balls have to be before we could ignore gravitational attraction between them? Could we make the balls that light?

Problem 4 Solution:



(a) To do this analysis we return to our methods of 8.01 and draw a free body diagram for the left mass, where the forces are gravity (down), electrostatic force (left) and tension in the string (up and to the left):



Since the ball is not accelerating, we know that the forces in both the horizontal and vertical directions must cancel:

$$F_y = T\cos\theta - F_g = 0 \Rightarrow T = \frac{F_g}{\cos\theta} = \frac{mg}{\cos\theta}$$

$$F_x = T\sin\theta - F_q = 0 \Rightarrow F_q = k\frac{q^2}{x^2} = T\sin\theta = \frac{mg\sin\theta}{\cos\theta} = mg\tan\theta \approx mg\frac{\frac{1}{2}x}{L}$$

where we have used the small angle approximation that $\tan\theta \approx \sin\theta$, which is valid when the length of the string L is long (relative to the separation x), which we are told it is. Continuing:

$$x^3 = k\frac{2q^2L}{mg} \quad \Rightarrow \quad \boxed{x = \left(\frac{2kq^2L}{mg}\right)^{1/3} = \left(\frac{q^2L}{2\pi\varepsilon_0 mg}\right)^{1/3}}$$

(b) This question is a little vague (intentionally). How small does something need to be so that we can ignore it? A 1% effect? 0.1%? It really depends on what we are trying to do. Let's just calculate for a fraction *f* of the electrostatic force, and we can go from there.

$$F_G \le fF_q \Rightarrow \frac{Gm^2}{x^2} \le f\frac{kq^2}{x^2} \Rightarrow m \le \sqrt{f\frac{kq^2}{G}}$$

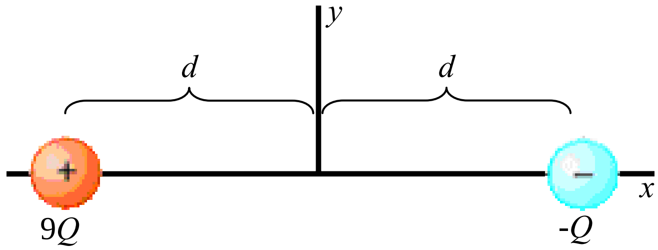
For a single electron charge we find the mass would need to obey:

$$m \le \sqrt{f\frac{\left(9\times10^9 \text{ Nm}^2\text{C}^{-2}\right)\left(1.6\times10^{-19} \text{ C}\right)^2}{\left(6.67\times10^{-11}\text{Nm}^2\text{kg}^{-2}\right)}} = \sqrt{f\cdot 3.5\times10^{-18}\text{kg}^2} = 1.9\times10^{-9} \text{ kg}\cdot\sqrt{\text{f}}$$

So, for a 1% effect ($f = 0.01$) we'd need $m \le 2\times10^{-9} \text{ kg}\cdot\sqrt{10^{-2}} = 2\times10^{-10} \text{ kg} = 200\,\text{ng}$

That is pretty small. We probably ***aren't justified*** in ignoring gravity in this case. But put on a million electrons so we are up near a gram and this becomes reasonable.

Problem 5:
Two massless point charges +9*Q* and -*Q* are fixed on the x-axis at x = -*d* and x = d:



(a) There is one point on the x-axis, x = x_0, where the electric field is zero. What is x_0?

(b) A third point charge q of mass m is free to move along the x-axis. What force does it feel if it is placed at x = x_0 (the location you just found)?

(c) Now q is displaced along the x-axis by a small distance a to the right. What sign of charge should q be so that it feels a force pulling it back to x = x_0?

(d) Show that if a is small compared to d ($a << d$) q will undergo simple harmonic motion. Determine the period of that motion. [NOTE: The motion of an object is simple harmonic if its acceleration is proportional to its position, but oppositely directed to the displacement from equilibrium. Mathematically, the equation of motion can be written as $d^2x/dt^2 = -\omega^2 x$, where ω is the angular frequency. See Review Module E for more detail. HINT: a/d is REALLY SMALL. Taylor expand]

(e) How fast will the charge q be moving when it is at the midpoint of its periodic motion?

Problem 5 Solution:

Two point charges 9*Q* and -*Q* are fixed on the x-axis at x = -*d* and x = d respectively.

(a) The charges are of opposite sign, so the field zero is not between them but to one side or the other. Since the 9*Q* charge at x = -*d* is bigger, the zero will be to the right of the -*Q* charge (x_0 > d). Now we can solve:

$$\vec{\mathbf{E}}(x_0) = \frac{1}{4\pi\varepsilon_0}\left(\frac{9Q}{(x_0+d)^2} + \frac{-Q}{(x_0-d)^2}\right) = 0 \quad \Rightarrow \quad \frac{9Q}{(x_0+d)^2} = \frac{Q}{(x_0-d)^2}$$

$$\Rightarrow \; 9(x_0-d)^2 = (x_0+d)^2 \;\; \Rightarrow \;\; 3(x_0-d) = (x_0+d) \;\; \Rightarrow \;\; \boxed{x_0 = 2d}$$

(b) Since $\vec{\mathbf{F}} = q\vec{\mathbf{E}}$ and $\vec{\mathbf{E}}(x_0) = 0$, the charge feels no force at x = x_0.

(c) To the right of x = x_0 = 2*d* the 9*Q* charge will dominate the -*Q* charge. You can see this most clearly if you go a far distance to the right. Looking back at the two charges they will appear as a single 8*Q* charge. In order to be attracted back to the net positive charge, $\boxed{q \text{ must be negative}}$.

(d) To show that the motion is simple harmonic we have to show that the acceleration of (and hence force on) *q* is proportional to its displacement. The force on it at x = 2*d* + *a* is:

$$F = \frac{q}{4\pi\varepsilon_0}\left(\frac{9Q}{(d+2d+a)^2} + \frac{-Q}{(d+a)^2}\right) = \frac{qQ}{4\pi\varepsilon_0}\left(\frac{9}{(3d+a)^2} + \frac{-1}{(d+a)^2}\right)$$

Now we have to make use of the approximation that $a \ll d$. The usual way to make use of such conditions is to make a small number out of the ratio ($a/d \ll 1$). So:

$$F = \frac{qQ}{4\pi\varepsilon_0 d^2}\left(\frac{1}{(1+a/3d)^2} + \frac{-1}{(1+a/d)^2}\right)$$

where we have factored out 3*d* from the first denominator and *d* from the second. Now we can do a Taylor expansion, keeping just the first order term: $(1+x)^{-2} \approx 1-2x$

$$F \approx \frac{qQ}{4\pi\varepsilon_0 d^2}\left((1-2a/3d)-(1-2a/d)\right) = \frac{qQ}{4\pi\varepsilon_0 d^2}\frac{4a}{3d} = \frac{qQ}{3\pi\varepsilon_0 d^3}a$$

Since $F = m\frac{d^2x}{dt^2}$, we have (at x = 2*d* + *a*)

$$\frac{d^2x}{dt^2} = -\frac{(-q)Q}{3\pi\varepsilon_0 m d^3}a$$

Since the acceleration is proportional to the displacement (*a*) but oppositely directed (recall that q is negative), we conclude that the motion of *q* is simple harmonic with

$$\omega^2 = \frac{|q|Q}{3\pi\varepsilon_0 m d^3}$$

The period is defined as $T = \frac{2\pi}{\omega}$, thus

$$\boxed{T = 2\pi\sqrt{\frac{3\pi\varepsilon_0 m d^3}{|q|Q}}}$$

(e) Since ω is the angular frequency of *q*, we can describe the position of *q*, $x(t)$, to be

$$x(t) = 2d + a\cos(\omega t)$$

To find the velocity, $v(t)$, we just differentiate *x* with respect to *t*:

$$v(t) = \frac{d}{dt}x(t) = -a\omega\sin\omega t$$

Since $-1 < \sin\omega t < 1$, the maximum speed is thus

$$\boxed{v_{\max} = a\omega = \omega^2 = a\sqrt{\frac{|q|Q}{3\pi\varepsilon_0 m d^3}}}$$

which is the speed of q at the midpoint of its motion (x = d).

Problem 6:

Consider two equal but opposite charges, both mass m, on the x-axis, $+Q$ at $(a,0)$ and $-Q$ at $(-a,0)$. They are connected by a rigid, massless, insulating rod whose center is fixed to a frictionless pivot at the origin. This is a dipole. A uniform field $\vec{\mathbf{E}} = E\hat{\mathbf{i}}$ is now applied.

(a) What is the force on the dipole due to this external field?

Now the dipole is rotated and held at a small angle θ_0 (c.c.w.) from the x-axis.

(b) Now what is the force on the dipole?

(c) How much did the potential energy of the dipole change when it was rotated?

(d) What is the torque on the dipole?

(e) Now the dipole is released and allowed to rotate due to this torque. Describe the motion that it undergoes (i.e. what is its angle $\theta(t)$?)

(f) Where is the positive charge when it is moving the fastest? How fast is it moving?

Problem 6 Solution:

(a) A dipole in a uniform field feels no force (the force on the positive charge is equal and opposite to the negative charge).

(b) The field is still uniform so the force is still zero.

(c) The change in potential energy of a charge q is given by:

$$\Delta U = q\Delta V = -q\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$$

For example, our positive charge changes its potential energy by:

$$\Delta U = -Q\int_{\theta=0}^{\theta_0} E\hat{\mathbf{i}} \cdot a\,d\theta\,\hat{\boldsymbol{\theta}} = -QEa\int_{\theta=0}^{\theta_0} d\theta\,\hat{\mathbf{i}} \cdot \left(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}\right) = QEa\int_{\theta=0}^{\theta_0} \sin\theta\,d\theta$$

$$= -QEa\cos\theta\Big|_0^{\theta_0} = QEa\left(1-\cos\theta_0\right)$$

Similarly, the negative charge moves from $\theta = \pi$ to $\theta = \pi + \theta_0$, changing its energy by:

$$\Delta U = -QEa\left(\cos\pi - \cos\left(\pi - \theta_0\right)\right) = -QEa\left(-1 + \cos\left(\theta_0\right)\right) = QEa\left(1-\cos\theta_0\right)$$

So both charges change their potential energy by the same, and the total change in the dipole's potential energy is: $\boxed{\Delta U_{dipole} = 2QEa\left(1-\cos\theta_0\right)}$

Note that we could write this as follows:

$$\Delta U_{dipole} = -2QEa\left(\cos\theta_0 - \cos 0\right) = -pE\left(\cos\theta_0 - \cos 0\right) = U_{final} - U_{init}$$

and hence the potential energy of a dipole is $U_{dipole} = -\vec{\mathbf{p}} \cdot \vec{\mathbf{E}}$

(d) From class we had that the torque is given by

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{p}} \times \vec{\mathbf{E}} \qquad \Rightarrow \qquad \boxed{\tau = pE\sin\theta_0}$$

Note that one can actually derive this from the potential energy. Just as the force is the derivative of the potential energy with respect to position, the torque is its derivative with respect to angle: $\tau = \frac{\partial}{\partial\theta}\left(-pE\cos\theta\right) = pE\sin\theta$

(e) Just thinking about it, the dipole wants to align with the field, so it will begin to rotate back towards $\theta = 0$. But since there is no friction it will overshoot but eventually turn around and come back. This is simple harmonic motion. Except now we are doing it for rotation. This is similar to last week's problem #7. We need to show that the torque depends linearly on the displacement angle: $\tau = pE\sin\theta \approx pE\theta$ where we have used the small angle approximation to linearize the sine. So we can write out the rotational equivalent of $F = ma$, $\tau \approx pE\theta = I\alpha = I\ddot{\theta}$, where I is the moment of inertia, $I = 2ma^2$.

So we have:

$$\ddot{\theta} = -\frac{pE}{I}\theta = -\frac{2qaE}{2ma^2}\theta = -\frac{qE}{ma}\theta$$

where I have had to stick in a minus sign because this is a restoring torque (it always causes rotation back towards $\theta = 0$). This is the same second order linear differential equation you wrote the answer to last week:

$$\theta(t) = \theta_0\cos(\omega t), \text{ where } \omega = \sqrt{\frac{qE}{ma}}$$

(f) The charge will be moving the fastest when it hits its equilibrium point. How do I know that? Its potential energy is a minimum there, so its kinetic energy must be a max. Plus, that is always the case in these simple harmonic motion problems. How fast is it going? We can get it easily by realizing that the charge is moving along an arc $\left(\Delta s = a\Delta\theta\right)$ and so its speed is given by:

$$v = \left|\frac{ds}{dt}\right| = a\left|\frac{d\theta}{dt}\right| = a\theta_0\omega\left|\sin(\omega t)\right|$$

Thus the maximum speed is where the sine = 1:

$$\boxed{\mathrm{v}_{\max} = a\theta_0\sqrt{\frac{qE}{ma}}}$$

Just because we always should, lets make sure this makes sense. If the charge is bigger it moves faster. If the arm is bigger is also moves faster (although it oscillates more slowly). That is subtle, so not particularly helpful as a dummy check. Bigger E means more v, that makes sense. So does the dependence on θ_0.

We also could have done this equating potential energy and kinetic energy (and realizing that both masses must be moving at the same speed):

$$\Delta U_{dipole} = 2QEa\left(1-\cos\theta_0\right) \approx 2QEa \cdot \tfrac{1}{2}\theta_0^2 = \text{Kinetic Energy} = \tfrac{1}{2}\left(2m\right)v^2$$

$$\Rightarrow \boxed{v = \theta_0\sqrt{\frac{QEa}{m}}}$$

where again we had to use a small angle approximation that $\cos\theta_0 \approx 1-\tfrac{1}{2}\theta_0^2$.