MATH 31A (Butler)
Practice for Final (A)

Try to answer the following questions without the use of book, notes or calculator; but you can use the equation sheet posted on the course website. Time yourself and try to finish the questions in less than three hours.

1. Find $\displaystyle\int_{-\pi/2}^{\pi} \sin|\theta|\,d\theta$.

2. (a) Find $\dfrac{d}{dx}(f(x)g(x)h(x))$ in terms of $f(x)$, $g(x)$, $h(x)$, $f'(x)$, $g'(x)$ and $h'(x)$.
(b) Starfleet intelligence has recently learned of a new threat. A new Borg vessel has been discovered that can change its shape, they are calling it the B-1000 (short for Borg-1000). The B-1000 though still has some limitations, first the only shape it can have is a three dimensional box and second the volume is always fixed, i.e., the box cannot deviate from a fixed volume.

You have been on a shuttle tracking the B-1000. From an earlier observation you saw that it had dimensions 20 meters by 15 meters by 10 meters. Currently though you can only see two sides. You notice that the length is currently 12 meters and is increasing at a rate of 1 meter per minute; the width is currently 25 meters and is decreasing at a rate of 2 meters per minute. What is the current depth of the B-1000 and how fast is it changing?

3. You and a classmate are preparing to give a presentation in your Astronomy 272 (stellar structure and evolution) course. You have decided that the best way to show how a star gets sucked into a black hole is through a modern interpretive dance where you will be playing the part of a large blue class O star and your partner will be the black hole. You will represent these two astronomical features using paper mache, and you are responsible for making your star. Initially you were planning to blow up a spherical balloon to a diameter of 16 inches before covering it in paper mache, but you ended up blowing the balloon to a diameter of 17 inches.

Using linear approximation get an estimate of how much more surface area the 17 inch balloon has as compared to the 16 inch balloon (i.e., an estimate of how much more paper you will need to make your model). (Hint: the surface area of a sphere of radius r is $4\pi r^2$.)

4. Find the area of the largest rectangle that can be placed into the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. (Assume that the sides of the rectangle are parallel to the two axes.)

5. Let $\ell(x) = \int_1^x \frac{1}{t}\,dt$.

(a) Find $\ell(1)$.

(b) Show that $\ell(ax) - \ell(x) = C$ where C is a constant. (Hint: a function is a constant if its derivative is 0.)

(c) Find the constant C for part (b). What can you conclude about $\ell(ax)$?

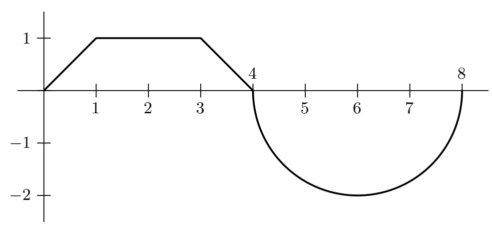
6. Reduce the following to a single integral of the form $A\int_B^C f(x)\,dx$ for some constants A, B, C.

$$\int_0^5 f(x)\,dx - \int_3^3 f(x^2)\,dx + \int_0^1 3f(3x)\,dx - \int_0^4 f(\frac{1}{2}x)\,dx + \int_5^3 f(x)\,dx.$$

7. For $a \neq -3, -2, -1$ find $\int (1 + \sqrt[3]{x})^a\,dx$. (Hint: try $u = 1 + \sqrt[3]{x}$.)

8. Let $g(x) = \int_2^x f(t)\,dt$ where $f(t)$ is the function defined piecewise by

$$f(t) = \begin{cases} t & \text{if } 0 \leq t \leq 1; \\ 1 & \text{if } 1 \leq t \leq 3; \\ 4 - t & \text{if } 3 \leq t \leq 4; \\ -\sqrt{4 - (t-6)^2} & \text{if } 4 \leq t \leq 8. \end{cases}$$



(a) Find the tangent line to $y = g(x)$ at $x = 6$.

(b) There is exactly one critical point for $g(x)$ in the interval $0 < x < 8$. Find the x value of the critical point and classify it as a maximum, minimum or neither using the first derivative test.

(c) Could we have classified the critical point found in part (b) using the second derivative test? Explain (briefly).

9. For $x \geq 1$ show

$$\arctan x - \frac{\pi}{4} \leq \int_1^x \frac{1}{t^2 + \cos^2 t}\,dt \leq 1 - \frac{1}{x}.$$

10. Prove that a volume of a cone with radius r and height h is $\frac{1}{3}\pi r^2 h$. (Hint: one way to make such a cone is to take the triangle with vertices at $(0, 0)$, $(r, 0)$ and (r, h) and rotate it around the x-axis.)