MATH 31A (Butler)
Practice for Midterm IIa (Solutions)

1. (a) Use linearization to give an estimate for $\sqrt[3]{1017}$.

We first note that $\sqrt[3]{1017}$ is close to $\sqrt[3]{1000}=10$ and so we can use linearization for the function $f(x)=\sqrt[3]{x}=x^{1/3}$ at $x=1000$. Before we start we note that $f'(x)=\frac{1}{3}x^{-2/3}=\frac{1}{3(\sqrt[3]{x})^2}$. So the linearization is

$$\begin{aligned} L(x) &= f(a)+f'(a)(x-a) \\ &= f(1000)+f'(1000)(x-1000) \\ &= \sqrt[3]{1000}+\frac{1}{3(\sqrt[3]{1000})^2}(x-1000) \\ &= 10+\frac{1}{300}(x-1000). \end{aligned}$$

So we have that

$$\sqrt[3]{1017}\approx L(1017)=10+\frac{1}{300}(1017-1000)=10+\frac{17}{300}.$$

(b) Is the estimate given in part (a) too large or too small? Explain.

We note that $f''(x)=-\frac{2}{9}x^{-5/3}$. In particular, at $x=1000$ we see that $f''(x)$ is negative and so the curve is concave down, i.e., it is bending down. So we would expect the actual function to be under the tangent line so that the answer in part (a) would be an overestimate.
[Note: $\sqrt[3]{1017}\approx 10.056348$ while $10+\frac{17}{300}\approx 10.056666$. So an overestimate as predicted, but still a good guess.]

2. What is the area of the largest rectangle that you can make where the bottom edge is on the x-axis and the top two vertices lie on the parabola $y = 12 - x^2$?

For this problem it is good to draw a picture (I recommend you draw one before proceeding with the argument). If one corner of the rectangle on the parabola is at the point $(x, y) = (x, 12 - x^2)$ (where we may assume that $x > 0$ then the remaining corners will be at $(x, 0)$ (the one directly below), $(-x, 12 - x^2)$ (the one directly to the left) and $(-x, 0)$ (the opposite corner). This rectangle has width $2x$ and height $12 - x^2$ so that the area of such a rectangle will be

$$A(x) = 2x(12 - x^2) = 24x - 2x^3.$$

We now maximize by finding the critical points, we have $A'(x) = 24 - 6x^2$ this is defined for all x and so the only critical point occurs when $A'(x) = 0$, i.e., $24 - 6x^2 = 0$ or $x^2 = 4$ or $x = 2$ (remember we assumed above that $x > 0$). Since $A''(x) = -12x$ we see at $x = 2$ that this has a negative second derivative and so is a maximum.

Therefore the width of the largest rectangle will be 4 and the height of the largest rectangle will be 8 for a total area of 32.

3. A new toy is sweeping the nation, *The Silly Putty Ellipse*. It looks like any old ellipse but you can change the shape and it will always stay an ellipse with the same area (it comes straight out of the box as a circle with a radius of 6 inches)!

Let a and b be the length of the two axis of the ellipse. As you are playing with the ellipse you notice at one point when $a = 3$ inches that you are increasing the length of a by a rate of $\frac{1}{8}$ of an inch per second. At that moment how fast is b changing (include units)?

(Hint: the area of an ellipse with the two axis with length a and b is πab.)

We have information about how a is changing, i.e., $\frac{da}{dt} = \frac{1}{8}$ and are trying to find information about $\frac{db}{dt}$. So this is clearly a related rates problem.
By the hint we have that

$$A = \pi ab.$$

Reading the problem we notice that the area of the ellipse out of the box is 36π (a circle with radius 6) and since the area never changes we can conclude that

$$36\pi = \pi ab \quad \text{or} \quad 36 = ab.$$

From this it is easy to see that when $a = 3$ that $b = 12$. Now taking the derivative with respect to t we get

$$0 = \frac{da}{dt}b + a\frac{db}{dt}.$$

Substituting what we know we have

$$0 = \frac{1}{8}12 + 3\frac{db}{dt} \quad \text{so} \quad \frac{db}{dt} = -\frac{1}{2}\ \frac{\text{inches}}{\text{second}}.$$

4. Consider the following function which is continuous and differentiable for all x (you do not need to prove this!):

$$f(x) = \begin{cases} x^3 + 2x^2 - 4x + 2 & \text{if } x \leq 1; \\ 3x - 2 & \text{if } x > 1. \end{cases}$$

For the interval $-4 \leq x \leq 2$ find all values of c that satisfies the Mean Value Theorem. (Hint: $f(-4) = -14$.)

First we recall the statement of the Mean Value Theorem. Namely, if our function is continuous and differentiable then between a and b there is some c so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

In our case we have $a = -4$ and $b = 2$ so that we are looking for *all* c so that

$$f'(c) = \frac{f(2) - f(-4)}{2 - (-4)} = \frac{4 - (-14)}{6} = \frac{18}{6} = 3.$$

The function given above is piecewise and so the derivative is also piecewise and so we have

$$f(x) = \begin{cases} 3x^2 + 4x - 4 & \text{if } x \leq 1; \\ 3 & \text{if } x > 1. \end{cases}$$

So we need to look in two places for our possible c. For $x \leq 1$ we need to have

$$f'(c) = 3c^2 + 4c - 4 = 3 \ \text{ or } \ 3c^2 + 4c - 7 = 0.$$

This factors, but we can also use the quadratic formula to get

$$c = \frac{-4 \pm \sqrt{4^2 - 4{\cdot}3{\cdot}(-7)}}{2{\cdot}3} = \frac{-4 \pm \sqrt{100}}{6} = \frac{-4 \pm 10}{6} = 1, -\frac{7}{3}.$$

Both of which are between -4 and 2. On the other hand for $x > 1$ we need to have

$$f'(c) = 3 = 3.$$

But this is always true! So all $x > 1$ that lie in the interval between -4 and 2 will also satisfy the result of the Mean Value Theorem.
Therefore the list of values c is

$$-\frac{7}{3} \text{ and } \textit{all} \text{ numbers between 1 and 2.}$$

5. The function $f(x) = \cos(x^3 - 2x)$ has a critical point at $x = 0$. Use the second derivative test to determine if it is a maximum or a minimum.

We need to compute the second derivative, so first let us compute the first derivative which we will do using the chain rule

$$f'(x) = -\big(\sin(x^3 - 2x)\big)\cdot(3x^2 - 2).$$

Now the second derivative is

$$f''(x) = -\big(\cos(x^3 - 2x)\big)\cdot(3x^2 - 2)^2 - \big(\sin(x^3 - 2x)\big)\cdot(6x).$$

We now evaluate at 0 to get

$$f''(0) = -\cos(0)\cdot(-2)^2 - \sin(0)\cdot 0 = -4.$$

So by the second derivative test we have that this is a local maximum.