MATH 31A (Butler)
Practice for Final (B)

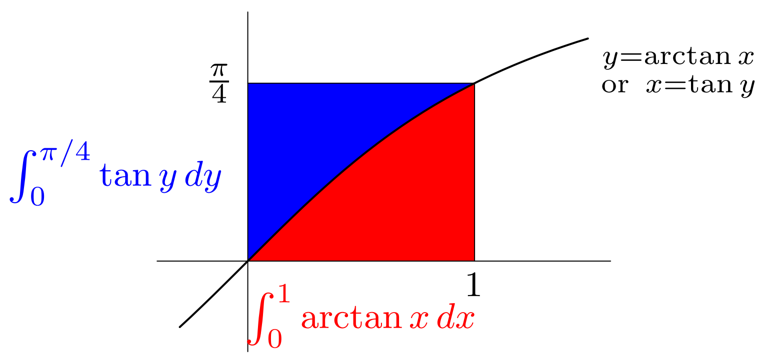
Try to answer the following questions without the use of book, notes or calculator; but you can use the equation sheet posted on the course website. Time yourself and try to finish the questions in less than three hours.

1. Show that $\displaystyle\int_0^{\pi/4} \tan x\,dx + \int_0^1 \arctan x\,dx = \frac{\pi}{4}$. (Hint: interpret each integral as an area and show how the areas "piece" together.)

First we note that we can change the "dummy" variables inside the integral. In particular, we are trying to find

$$\int_0^{\pi/4} \tan y\,dy + \int_0^1 \arctan x\,dx.$$

But since $x = \tan y$ if and only if $y = \arctan x$ (for x and y in this range), then these two integrals both involve the same curve. In particular, these integrals correspond to the respective areas shown below.



Since the two areas piece together to form a rectangle with dimensions 1 and $\frac{\pi}{4}$ then the combined area is the area of the rectangle which is $\frac{\pi}{4}$.

2. A particle moves along the curve implicitly defined by $xy^4 - yx^4 = x - y^2$. When the particle passes through the point $(1, 1)$ its x coordinate is changing $1/4$ units per second. How fast is the y coordinate changing?

We have $x = 1$, $y = 1$ and $\frac{dx}{dt} = \frac{1}{4}$ and we are looking for $\frac{dy}{dt}$. This is a related rates problem and so we take the derivative of both sides of the implicitly defined function with respect to t. So we have

$$\frac{dx}{dt}\cdot y^4 + x\cdot 4y^3\frac{dy}{dt} - \frac{dy}{dt}\cdot x^4 - y\cdot 4x^3\frac{dx}{dt} = \frac{dx}{dt} - 2y\frac{dy}{dt}.$$

Substituting what we know we have

$$\frac{1}{4}\cdot 1^4 + 1\cdot 4\cdot 1^3\frac{dy}{dt} - \frac{dy}{dt}\cdot 1^4 - 1\cdot 4\cdot 1^3\cdot\frac{1}{4} = \frac{1}{4} - 2\cdot 1\cdot\frac{dy}{dt},$$

which rearranges to

$$(4 - 1 + 2)\frac{dy}{dt} = \frac{1}{4} - \frac{1}{4} + 1 \quad \text{or} \quad 5\frac{dy}{dt} = 1 \quad \text{or} \quad \frac{dy}{dt} = \frac{1}{5}\,\frac{\text{units}}{\text{sec}}.$$

3. The night before the final you have decided to do one last study session. But before you begin you decide that you want to make the best use of your time. You know that there is a diminishing return to the amount of time you study (i.e., you get more out of your first hour of study than you will your second; and more out of your second hour than you will your third). At the same time you know that the longer you study the less sleep you will have and the harder it will be to concentrate on the test (which will make problems about optimizing your study session even harder!). After thinking about it for a few minutes you decide if H is the number of hours that you study that night then you anticipate your score on the final will be

$$115 - \frac{50}{H+1} - 8H,$$

more or less. How many hours should you study to maximize your score, and what should your anticipated score on the final be?

This is clearly an optimization problem since we are trying to maximize our score. So in this case we will take the derivative and set it equal to 0 (note that clearly we must have $H \geq 0$ since we cannot go back and study less in the past). So taking the derivative of our anticipated score we have

$$\frac{d}{dH}\left(115 - \frac{50}{H+1} - 8H\right) = \frac{50}{(H+1)^2} - 8.$$

For $H \geq 0$ this is never undefined so our only critical point will be when the derivative is 0, which happens when

$$\frac{50}{(H+1)^2} = 8 \quad \text{or} \quad (H+1)^2 = \frac{50}{8} = \frac{25}{4} \quad \text{or} \quad H+1 = \frac{5}{2} \quad \text{so} \quad H = \frac{3}{2} \text{ hrs.}$$

We only found one critical point and so that must be the right answer (unless somehow studying this length of time is the way to minimize our score!) But to double check by taking the second derivative we get $-100/(H+1)^3$ which is clearly negative at $H = 3/2$ so that by the second derivative test this is a maximum.

So we should study for $H = 3/2$ hours, or an hour and a half, and our anticipated score on the final will be

$$115 - \frac{50}{(3/2)+1} - 8\cdot\frac{3}{2} = 115 - \frac{50}{5/2} - 12 = 115 - 20 - 12 = 83.$$

Now if we can only get the graders to give us that score we would be happy!

4. Use the following information to get an estimate for $g(f(2.1))$.

x	0	1	2	3
$f(x)$	-1	3	0	2
$f'(x)$	1	-2	3	0
$g(x)$	3	0	1	2
$g'(x)$	1	3	2	2

It seems reasonable to thing that $g(f(2.1))$ should be close to $g(f(2))$. So let us find the tangent line to $h(x) = g(f(x))$ at $x = 2$ and evaluate it at $x = 2.1$ to get our approximation. To do this we note that $h'(x) = g'(f(x))f'(x)$ and so

$$\begin{aligned} h(2) &= g(f(2)) = g(0) = 3, \quad \text{and} \\ h'(2) &= g'(f(2))f'(2) = g'(0)f'(2) = 1 \cdot 3 = 3. \end{aligned}$$

So we have that the tangent line to $h(x)$ at $x = 2$ is

$$y = 3 + 3(x - 2) = 3x - 3.$$

Therefore plugging in $x = 2.1$ we have

$$g(f(2.1)) = h(2.1) \approx 3 \cdot 2.1 - 3 = 3.3.$$

5. (a) Find $\int \left(\sec\theta + \tan\theta\right)^2 \, d\theta$.

We first work on rewriting the integral. Expanding we have

$$\int \left(\sec\theta + \tan\theta\right)^2 d\theta = \int \left(\sec^2\theta + 2\sec\theta\tan\theta + \tan^2\theta\right) d\theta.$$

The $\sec^2\theta$ term and the $2\sec\theta\tan\theta$ term are both easy to integrate, but we don't have a way to integrate the $\tan^2\theta$ term. But that is no problem because we can substitute $\sec^2\theta - 1$ for $\tan^2\theta$. And so we have

$$\begin{aligned}\int \left(\sec^2\theta + 2\sec\theta\tan\theta + \tan^2\theta\right) d\theta \\ &= \int \left(\sec^2\theta + 2\sec\theta\tan\theta + (\sec^2\theta - 1)\right) d\theta \\ &= \int \left(2\sec^2\theta + 2\sec\theta\tan\theta - 1\right) d\theta = 2\tan\theta + 2\sec\theta - \theta + C.\end{aligned}$$

(b) Find $\int \frac{x}{\sqrt{x^2+1}+x} \, dx$.

To start let us multiply top and bottom by the conjugate of the bottom (this will help to simplify the denominator). So we have

$$\begin{aligned}\int \frac{x}{\sqrt{x^2+1}+x}\,dx &= \int \frac{x}{\sqrt{x^2+1}+x}\cdot\frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}-x}\,dx \\ &= \int \frac{x\sqrt{x^2+1}-x^2}{(x^2+1)-x^2}\,dx = \int \left(x\sqrt{x^2+1}-x^2\right)dx \\ &= \int x\sqrt{x^2+1}\,dx - \int x^2\,dx = \int \underbrace{x\sqrt{x^2+1}}_{\substack{u=x^2+1 \\ du=2dx}}\,dx - \frac{1}{3}x^3 + C \\ &= \int u^{1/2}\cdot\frac{1}{2}\,du - \frac{1}{3}x^3 + C = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} - \frac{1}{3}x^3 + C \\ &= \frac{1}{3}(x^2+1)^{3/2} - \frac{1}{3}x^3 + C.\end{aligned}$$

We also needed to use substitution in the middle. Note that in this case we broke the integral up over addition *before* we did the substitution. This was because the same substitution would have gotten us into more trouble had we tried to apply it to the integral of x^2, besides which the integral of x^2 is easy enough to do by itself!

6. Suppose that $y''(t) = 2 - \sin(\pi t)$ and that $y(0) = -2$ and $y(2) = 5$. What is $y'(1)$?

Let us try and construct $y'(t)$. We know that

$$y'(t) = \int y''(t)\,dt + C = \int \big(2 - \sin(\pi t)\big)\,dt + C = 2t + \frac{1}{\pi}\cos(\pi t) + C.$$

Now if we know the constant C we would be done, but we don't have any information about y' to use to determine C, what we do have is information about y so let us now turn to finding y. So we similarly have

$$y(t) = \int y'(t)\,dt + D = \int \Big(2t + \frac{1}{\pi}\cos(\pi t) + C\Big)\,dt + D$$
$$= t^2 + \frac{1}{\pi^2}\sin(\pi t) + Ct + D.$$

(Note that D is a constant but it is arbitrary and can be different from C, hence we used a different letter to denote this different constant.) Now we do have information about y, in particular we can use the two known values of y to solve for the two constants C and D. So we have

$$\begin{aligned} -2 &= y(0) = D, \\ 5 &= y(2) = 4 + 2C + D. \end{aligned}$$

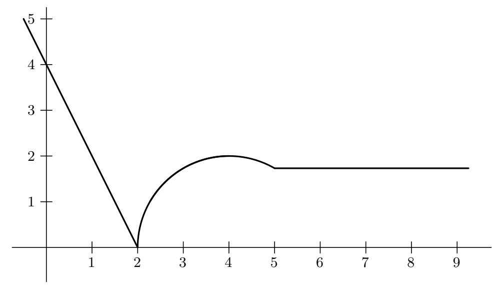
From these we see that $D = -2$ and then $2C = 3$ or $C = \dfrac{3}{2}$. Therefore we have

$$y'(1) = 2 + \frac{1}{\pi}\cos\pi + C = 2 - \frac{1}{\pi} + \frac{3}{2} = \frac{7}{2} - \frac{1}{\pi}.$$

(Note: this problem is similar in spirit to problems in differential equations where we know what the function is on the *boundary* and we try to figure out what the function is doing on the *interior* given that it satisfies some relationship with its derivatives. This turns out to be important for physics and many other problems.)

7. Let $H(x) = \int_{x^2}^{x+2} h(t)\,dt$ where $h(t)$ is the function defined piecewise by

$$h(t) = \begin{cases} 4-2t & \text{if } t \le 2, \\ \sqrt{4-(t-4)^2} & \text{if } 2 \le t \le 5, \\ \sqrt{3} & \text{if } t \ge 5. \end{cases}$$



A graph of this function is shown on the right. Find the following values.

(a) $H(-2) =$

Using the area of a triangle, the area of one fourth of a circle, and basic properties of the integral we have

$$H(-2) = \int_4^0 h(t)\,dt = -\int_0^4 h(t)\,dt = -\left(\frac{1}{2}\cdot 2\cdot 4 + \frac{1}{4}\cdot\pi\cdot 2^2\right) = -4-\pi.$$

(b) $H(0) =$

Using the area formula for a triangle, we have

$$H(0) = \int_0^2 h(t)\,dt = \frac{1}{2}\cdot 2\cdot 4 = 4.$$

(c) $H'(1) =$

By the FTC we have that $H'(x) = h(x+2)\cdot 1 - h(x^2)\cdot 2x$, so we have

$$H'(1) = h(3) - 2h(1) = \sqrt{3} - 2\cdot 2 = \sqrt{3} - 4.$$

(d) $H(2) =$

Using basic properties of integration we have

$$H(2) = \int_4^4 h(t)\,dt = 0.$$

(e) $H(3) =$

Using the area formula for a rectangle, we have

$$H(3) = \int_9^5 h(t)\,dt = -\int_5^9 h(t)\,dt = -4\sqrt{3}.$$

8. What is the average area of circles where the radii of the circles can range from 1 to 4?

The area of a circle with radius r is πr^2. Therefore to find the average area of circles with radii between 1 and 4 is the same as finding the average of πr^2 with $1 \le r \le 4$ which is

$$\text{Average} = \frac{1}{4-1}\int_1^4 \pi r^2\,dr = \frac{\pi}{3}\cdot\frac{1}{3}x^3\Big|_{r=1}^{r=4} = \frac{\pi}{9}\big(4^3 - 1^3\big) = \frac{\pi}{9}(64-1) = 7\pi.$$

9. Let $h(x)$ be a function such that

$$\int_0^2 h(x)\,dx = 1,\ \int_0^3 h(x)\,dx = 2,\ \int_0^4 h(x)\,dx = 6,\ \int_1^5 h(x)\,dx = 5,\ \text{and}\ \int_2^5 h(x)\,dx = 7.$$

Find $\displaystyle\int_1^3 h(x)\,dx$.

Our basic approach is to combine these integrals for which we have a value in various ways to get information about the integral we are interested in. One approach is as follows:

$$\int_0^5 h(x)\,dx = \int_0^2 h(x)\,dx + \int_2^5 h(x)\,dx = 1 + 7 = 8.$$

Using this we have

$$\int_0^1 h(x)\,dx = \int_0^5 h(x)\,dx - \int_1^5 h(x)\,dx = 8 - 5 = 3,$$

So finally we have

$$\int_1^3 h(x)\,dx = \int_0^3 h(x)\,dx - \int_0^1 h(x)\,dx = 2 - 3 = -1.$$

(Note that we did not need to use all of the information given to us.)

10. A torus (or as Homer Simpson would say "mmmmm, donut") can be formed by spinning a circle around the x-axis. Find the volume of the torus found by spinning the circle of radius 1 centered at $(0, 2)$ around the x-axis.
(Hint: the curve describing the top of the circle is $y = 2 + \sqrt{1 - x^2}$ while the curve describing the bottom of the circle is $y = 2 - \sqrt{1 - x^2}$.)

This is finding a solid of revolution, for which we have nice formulas for. In particular we have

$$\begin{aligned}
\text{Volume} &= \pi \int_{-1}^{1} \left(\left(2 + \sqrt{1 - x^2}\right)^2 - \left(2 - \sqrt{1 - x^2}\right)^2 \right) dx \\
&= \pi \int_{-1}^{1} \left(\left(4+4\sqrt{1-x^2}+(1-x^2)\right) - \left(4-4\sqrt{1-x^2}+(1-x^2)\right) \right) dx \\
&= 8\pi \int_{-1}^{1} \sqrt{1 - x^2}\, dx \\
&= 8\pi \cdot \frac{\pi}{2} = 4\pi^2.
\end{aligned}$$

(Or as Homer would say "four square pies, mmmmm".) For the last step we used that $\int_{-1}^{1} \sqrt{1 - x^2}\, dx$ is the area of half of a circle of radius 1, so has value $\pi/2$.