22 The z-Transform

Recommended Problems

P22.1

An LTI system has an impulse response $h[n]$ for which the z-transform is

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

(a) Plot the pole-zero pattern for $H(z)$.

(b) Using the fact that signals of the form $z^n$ are eigenfunctions of LTI systems, determine the system output for all $n$ if the input $x[n]$ is

$$x[n] = (\frac{3}{2})^n + 3(2)^n$$

P22.2

Consider the sequence $x[n] = 2^nu[n]$.

(a) Is $x[n]$ absolutely summable?

(b) Does the Fourier transform of $x[n]$ converge?

(c) For what range of values of $r$ does the Fourier transform of the sequence $r^{-n}x[n]$ converge?

(d) Determine the z-transform $X(z)$ of $x[n]$, including a specification of the ROC.

(e) $X(z)$ for $z = 3e^{j\theta}$ can be thought of as the Fourier transform of a sequence $x_r[n]$, i.e.,

$$2^ru[n] \xrightarrow{Z} X(z),$$

$$x_r[n] \xrightarrow{F} X(3e^{j\theta}) = X_x(e^{j\theta})$$

Determine $x_r[n]$.

P22.3

Shown in Figure P22.3 is the pole-zero plot for the z-transform $X(z)$ of a sequence $x[n]$.

![Figure P22.3](image-url)
Determine what can be inferred about the associated region of convergence from each of the following statements.

(a) \( x[n] \) is right-sided.
(b) The Fourier transform of \( x[n] \) converges.
(c) The Fourier transform of \( x[n] \) does not converge.
(d) \( x[n] \) is left-sided.

P22.4

(a) Determine the \( z \)-transforms of the following two signals. Note that the \( z \)-transforms for both have the same algebraic expression and differ only in the ROC.

(i) \( x_1[n] = (\frac{1}{2})^n u[n] \)
(ii) \( x_2[n] = -(\frac{1}{2})^n u[-n - 1] \)

(b) Sketch the pole-zero plot and ROC for each signal in part (a).
(c) Repeat parts (a) and (b) for the following two signals:

(i) \( x_3[n] = 2u[n] \)
(ii) \( x_4[n] = -(2)^n u[-n - 1] \)

(d) For which of the four signals \( x_1[n], x_2[n], x_3[n], \) and \( x_4[n] \) in parts (a) and (c) does the Fourier transform converge?

P22.5

Consider the pole-zero plot of \( H(z) \) given in Figure P22.5, where \( H(a/2) = 1 \).

(a) Sketch \( |H(e^{j\omega})| \) as the number of zeros at \( z = 0 \) increases from 1 to 5.
(b) Does the number of zeros affect \( \angle H(e^{j\omega}) \)? If so, specifically in what way?
(c) Find the region of the \( z \) plane where \( |H(z)| = 1 \).
Determine the $z$-transform (including the ROC) of the following sequences. Also sketch the pole-zero plots and indicate the ROC on your sketch.

(a) $(\frac{1}{2})^n u[n]$
(b) $\delta[n + 1]$

For each of the following $z$-transforms determine the inverse $z$-transform.

(a) $X(z) = \frac{1}{1 + \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$
(b) $X(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-2}}, \quad |z| > \frac{1}{2}$
(c) $X(z) = \frac{1 - az^{-1}}{z^{-1} - a}, \quad |z| > \left| \frac{1}{a} \right|$

Optional Problems

In this problem we study the relation between the $z$-transform, the Fourier transform, and the ROC.

(a) Consider the signal $x[n] = u[n]$. For which values of $r$ does $r^{-n}x[n]$ have a converging Fourier transform?
(b) In the lecture, we discussed the relation between $X(z)$ and $\mathcal{F}[r^{-n}x[n]]$. For each of the following values of $r$, sketch where in the $z$ plane $X(z)$ equals the Fourier transform of $r^{-n}x[n]$.
   (i) $r = 1$
   (ii) $r = \frac{1}{2}$
   (iii) $r = 3$
(c) From your observations in parts (a) and (b), sketch the ROC of the $z$-transform of $u[n]$.

Suppose $X(z)$ on the circle $z = 2e^{j\theta}$ is given by

$$X(2e^{j\theta}) = \frac{1}{1 - \frac{1}{4}e^{-j\theta}}$$

Using the relation $X(re^{j\theta}) = \mathcal{F}[r^{-n}x[n]]$, find $2^{-n}x[n]$ and then $x[n]$, the inverse $z$-transform of $X(z)$.

(b) Find $x[n]$ from $X(z)$ below using partial fraction expansion, where $x[n]$ is known to be causal, i.e., $x[n] = 0$ for $n < 0$.

$$X(z) = \frac{3 + 2z^{-1}}{2 + 3z^{-1} + z^{-2}}$$
P22.10

A discrete-time system with the pole-zero pattern shown in Figure P22.10-1 is referred to as a first-order all-pass system because the magnitude of the frequency response is a constant, independent of frequency.

(a) Demonstrate algebraically that $|H(e^{j\omega})|$ is constant.

(b) To demonstrate the same property geometrically, consider the vector diagram in Figure P22.10-2. Show that the length of $v_2$ is proportional to the length of $v_1$ independent of $\Omega$ by following these two steps:

(i) Express the length of $v_1$ using the law of cosines and the fact that it is one leg of a triangle for which the other two legs are the unit vector and a vector of length $a$.

(ii) In a manner similar to that in step (i), determine the length of $v_2$ and show that it is proportional in length to $v_1$ independent of $\Omega$.

P22.11

Parts (a)–(e) (Figures P22.11-1 to P22.11-5) give pole-zero plots, and parts (i)–(iv) (Figures P22.11-6 to P22.11-9) give sketches of possible Fourier transform magnitudes. Assume that for all the pole-zero plots, the ROC includes the unit circle. For each pole-zero plot (a)–(e), specify which one if any of the sketches (i)–(iv) could represent the associated Fourier transform magnitude. More than one pole-zero plot may be associated with the same sketch.
The z-Transform

Problems

(a) Figure P22.11-1

(b) Figure P22.11-2

(c) Figure P22.11-3

(d) Figure P22.11-4

(e) Figure P22.11-5

(i) Figure P22.11-6

(ii) Figure P22.11-7
Determine the $z$-transform for the following sequences. Express all sums in closed form. Sketch the pole-zero plot and indicate the ROC. Indicate whether the Fourier transform of the sequence exists.

(a) $(\frac{1}{2})^n[u[n] - u[n - 10]]$
(b) $(\frac{1}{3})^n$
(c) $7 \left(\frac{1}{3}\right)^n \cos \left[\frac{2\pi n}{6} + \frac{\pi}{4}\right] u[n]$
(d) $x[n] = \begin{cases} 0, & n < 0 \\ 1, & 0 \leq n \leq 9 \\ 0, & 9 < n \end{cases}$

Using the power-series expansion

$$
\log(1 - w) = -\sum_{i=1}^{\infty} \frac{w^i}{i}, \quad |w| < 1,
$$
determine the inverse of the following $z$-transforms.

(a) $X(z) = \log(1 - 2z), \quad |z| < \frac{1}{2}$
(b) $X(z) = \log(1 - \frac{1}{2}z^{-1}), \quad |z| > \frac{1}{2}$
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