20 The Laplace Transform

Recommended Problems

P20.1

Consider the signal \( x(t) = 3e^{2t}u(t) + 4e^{3t}u(t) \).

(a) Does the Fourier transform of this signal converge?

(b) For which of the following values of \( \sigma \) does the Fourier transform of \( x(t)e^{-\sigma t} \) converge?

(i) \( \sigma = 1 \)

(ii) \( \sigma = 2.5 \)

(iii) \( \sigma = 3.5 \)

(c) Determine the Laplace transform \( X(s) \) of \( x(t) \). Sketch the location of the poles and zeros of \( X(s) \) and the ROC.

P20.2

Determine the Laplace transform, pole and zero locations, and associated ROC for each of the following time functions.

(a) \( e^{-at}u(t), \quad a > 0 \)

(b) \( e^{-at}u(t), \quad a < 0 \)

(c) \( -e^{-at}u(-t), \quad a < 0 \)

P20.3

Shown in Figures P20.3-1 to P20.3-4 are four pole-zero plots. For each statement in Table P20.3 about the associated time function \( x(t) \), fill in the table with the corresponding constraint on the ROC.

(a) 

(b)
### P20.3

**Figure P20.3-3**

**Figure P20.3-4**

### Constraint on ROC for Pole-Zero Pattern

<table>
<thead>
<tr>
<th>$x(t)$</th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>(d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) Fourier transform of $x(t)e^{-t}$ converges</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(ii) $x(t) = 0$, $t &gt; 10$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(iii) $x(t) = 0$, $t &lt; 0$</td>
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</tbody>
</table>

**Table P20.3**

### P20.4

Determine $x(t)$ for the following conditions if $X(s)$ is given by

$$X(s) = \frac{1}{(s + 1)(s + 2)}$$

- **(a)** $x(t)$ is right-sided
- **(b)** $x(t)$ is left-sided
- **(c)** $x(t)$ is two-sided

### P20.5

An LTI system has an impulse response $h(t)$ for which the Laplace transform $H(s)$ is

$$H(s) = \int_{-\infty}^{+\infty} h(t)e^{-st}dt = \frac{1}{s + 1}, \quad Re(s) > -1$$

Determine the system output $y(t)$ for all $t$ if the input $x(t)$ is given by

$$x(t) = e^{-t/2} + 2e^{-t/3} \quad \text{for all } t.$$
P20.6

(a) From the expression for the Laplace transform of \( x(t) \), derive the fact that the Laplace transform of \( x(t) \) is the Fourier transform of \( x(t) \) weighted by an exponential.

(b) Derive the expression for the inverse Laplace transform using the Fourier transform synthesis equation.

Optional Problems

P20.7

Determine the time function \( x(t) \) for each Laplace transform \( X(s) \).

(a) \( \frac{1}{s + 1}, \quad Re[s] > -1 \)

(b) \( \frac{1}{s + 1}, \quad Re[s] < -1 \)

(c) \( \frac{s}{s^2 + 4}, \quad Re[s] > 0 \)

(d) \( \frac{s + 1}{s^2 + 5s + 6}, \quad Re[s] > -2 \)

(e) \( \frac{s + 1}{s^2 + 5s + 6}, \quad Re[s] < -3 \)

(f) \( \frac{s^2 - s + 1}{s^2(s - 1)}, \quad 0 < Re[s] < 1 \)

(g) \( \frac{s^2 - s + 1}{(s + 1)^2}, \quad -1 < Re[s] \)

(h) \( \frac{s + 1}{(s + 1)^2 + 4}, \quad Re[s] > -1 \)

Hint: Use the result from part (c).

P20.8

The Laplace transform \( X(s) \) of a signal \( x(t) \) has four poles and an unknown number of zeros. \( x(t) \) is known to have an impulse at \( t = 0 \). Determine what information, if any, this provides about the number of zeros.

P20.9

Determine the Laplace transform, pole-zero location, and associated ROC for each of the following time functions.

(a) \( e^{-at}u(t), \quad a < 0 \)

(b) \( -e^{at}u(-t), \quad a > 0 \)

(c) \( e^{at}u(t), \quad a > 0 \)

(d) \( e^{-a|t|}, \quad a > 0 \)
(e) \( u(t) \)
(f) \( \delta(t - t_0) \)
(g) \( \sum_{k=0}^{\infty} a^k \delta(t - kT), \quad a > 0 \)
(h) \( \cos(\omega_0 t + b)u(t) \)
(i) \( \sin(\omega_0 t + b)e^{-at}u(t), \quad a > 0 \)

**P20.10**

(a) If \( x(t) \) is an even time function such that \( x(t) = x(-t) \), show that this requires that \( X(s) = X(-s) \).
(b) If \( x(t) \) is an odd time function such that \( x(t) = -x(-t) \), show that \( X(s) = -X(-s) \).
(c) Determine which, if any, of the pole-zero plots in Figures P20.10-1 to P20.10-4 could correspond to an even time function. For those that could, indicate the required ROC.

(1) 

(2) 

Figure P20.10-1

(3) 

(4) 

Figure P20.10-2

(iii) 

(iv) 

Figure P20.10-3

Figure P20.10-4

(d) Determine which, if any, of the pole-zero plots in part (c) could correspond to an odd time function. For those that could, indicate the required ROC.