25 Feedback

Solutions to Recommended Problems

S25.1

We have

\[ V(s) = X(s) - Y(s)K(s) \]  \hspace{1cm} (S25.1-1)

and

\[ Y(s) = V(s)H(s) \] \hspace{1cm} (S25.1-2)

From eq. (S25.1-2),

\[ V(s) = \frac{Y(s)}{H(s)} \] \hspace{1cm} (S25.1-3)

Substituting eq. (S25.1-3) into eq. (S25.1-1), we have

\[
\frac{Y(s)}{H(s)} = X(s) - Y(s)K(s),
\]

\[
Y(s)[1 + H(s)K(s)] = H(s)X(s),
\]

\[
\frac{Y(s)}{X(s)} = \frac{H(s)}{1 + H(s)K(s)}
\]

Similarly,

\[
\frac{Y(z)}{X(z)} = \frac{H(z)}{1 + H(z)K(z)}
\]

(b) \[ Q(s) = \frac{H(s)}{1 + KH(s)} \], \[ Q(z) = \frac{H(z)}{1 + KH(z)} \]

For \( H(s) = 2/(s - 2) \) and \( H(z) = 2/(z - 2) \),

\[
Q(s) = \frac{2}{(s - 2) + 2K} = \frac{2}{s - 2(1 - K)}
\]

\[
Q(z) = \frac{2}{(z - 2) + 2K} = \frac{2}{z - 2(1 - K)}
\]

For \( K = 0 \),

\[
Q(s) = \frac{2}{s - 2} \quad \text{and} \quad Q(z) = \frac{2}{z - 2}
\]

as shown in Figures S25.1-2 and S25.1-3, respectively.
For $K = -1$,

$$Q(s) = \frac{2}{s - 4} \quad \text{and} \quad Q(z) = \frac{2}{z - 4},$$

as shown in Figures S25.1-4 and S25.1-5, respectively.

For $K = 1$,

$$Q(s) = \frac{2}{s} \quad \text{and} \quad Q(z) = \frac{2}{z},$$

as shown in Figures S25.1-6 and S25.1-7, respectively.

(c) $Q(s) = \frac{2}{s - 2(1 - K)}$

The pole is located at $s = 2(1 - K)$, as shown in Figure S25.1-8.
Hence, the locus of the pole is the line $Re{s} = 0$. Similarly, for

$$Q(z) = \frac{2}{z - 2(1 - K)},$$

the locus of the pole is also the line $Re{z} = 0$, shown in Figure S25.1-9.

The root location decreases as $K$ moves to infinity and increases as $K$ moves to negative infinity.

(d) $Q(s) = \frac{2}{s - 2(1 - K)}$

The system is stable for $2(1 - K) < 0$, or $K > 1$.

$$Q(z) = \frac{2}{z - 2(1 - K)}$$

The system is stable for $-1 < 2(1 - K) < 1$, or $\frac{1}{2} < K < \frac{3}{2}$.

S25.2

We use Problem P25.1.

(a) (i) $\frac{Y(s)}{X(s)} = \frac{H(s)}{1 + G(s)H(s)}$

(ii) $E(s) = X(s) - R(s)$

$= X(s) - Y(s)G(s)$

$= X(s) - E(s)H(s)G(s)$,

$E(s)[1 + H(s)G(s)] = X(s)$,

$\frac{E(s)}{X(s)} = \frac{1}{1 + H(s)G(s)}$
(iii) \[ \frac{Y(s)}{E(s)} = H(s) \]

(iv) \[ \frac{Y(s)}{R(s)} = \frac{1}{G(s)} \]

(b) \[ W(z) = X(z) \frac{H_1(z)}{1 + G(z)H_1(z)} \]
\[ Y(z) = W(z) + X(z)H_0(z) \]
\[ Y(z) = \frac{X(z)H_1(z)}{1 + G(z)H_1(z)} + X(z)H_0(z) \]

Thus,
\[ \frac{Y(z)}{X(z)} = \frac{H_1(z)}{1 + G(z)H_1(z)} + H_0(z) \]

(c) \[ \frac{Y(s)}{W(s)} = \frac{H_1(s)}{1 + G_1(s)H(s)} \], as shown in Figure S25.2.

\[
\frac{Y(s)}{X(s)} = \frac{H_1(s)H_0(s)}{1 + G_1(s)H_1(s)}
\]
\[
= \frac{H_1(s)H_0(s)}{1 + G_1(s)H_0(s) + G_0(s)H_1(s)H_0(s)}
\]

S25.3

(a)
(b) From the frequency response in part (a), clearly system 1 tends to make the response more constant and system 2 tends to resemble the inverse of $G(j\omega)$.

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S25.4

For the system in Figure S25.4-1, we denote the closed-loop system function by

$$V = \frac{H}{1 + GH}$$

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(a) $V(s) = \frac{1}{(s + 1)(s + 3)} = \frac{1}{(s + 1)(s + 3) + 1}$

Therefore,

$$v(t) = te^{-2t}u(t)$$

(b) $V(s) = \frac{\frac{1}{s + 3}}{1 + \frac{1}{s + 3}} = \frac{1}{(s + 3) + (s + 1)}$

$$= \frac{1}{2s + 4} = \frac{1}{2}\frac{1}{s + 2}$$

In this case,

$$v(t) = \frac{1}{2}e^{-2t}u(t)$$

(c) The system function $G(s) = e^{-s/3}$ corresponds to a delay of $\frac{1}{3}$, i.e., the feedback system of Figure P25.4(a) becomes that shown in Figure S25.4-2.
We can now recursively obtain the impulse response by inspection. With $x(t) = \delta(t)$,

$$y(t) = \frac{1}{2}\delta(t) - \frac{1}{6}\delta(t - \frac{1}{3}) + \frac{1}{6}\delta(t - \frac{1}{3}) - \cdots$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n \delta\left(t - \frac{n}{3}\right)$$

\[\text{(d)}\]

$$V(z) = \frac{z^{-1}}{1 - \frac{1}{3}z^{-1}}$$

\[\text{(e)}\]

$$V(z) = \frac{H(z)}{1 + H(z)G(z)} = \frac{\frac{1}{2} - \frac{1}{3}z^{-1}}{1 + \left(\frac{2}{3} - \frac{1}{6}z^{-1}\right)\left(\frac{z^{-1}}{1 - \frac{1}{3}z^{-1}}\right)}$$

$$= \frac{\left(\frac{3}{2} - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)}{(1 - \frac{1}{3}z^{-1}) + \left(\frac{3}{2} - \frac{1}{2}z^{-1}\right)z^{-1}}$$

$$= \frac{\frac{3}{2} - \frac{1}{2}z^{-1} + \frac{1}{2}z^{-2}}{1 + \frac{1}{3}z^{-1} - \frac{1}{3}z^{-2}}$$

Therefore,

$$v[n] = \frac{3}{2}(\tfrac{1}{3})^nu[n] - \left(-\tfrac{1}{3}\right)^nu[n]$$

Thus,

$$v[n] = \frac{3}{2}\delta[n + 1] - \frac{3}{2}\delta[n] + \frac{1}{2}\delta[n - 1],$$

where $\delta[n]$ is $v[n]$ in part (d).
### Solutions to Optional Problems

#### S25.5

\[ y(t) = K_2w(t) + K_1K_2v(t) \quad (\text{S25.5-1}) \]

By taking the transform of eq. (S25.5-1), we have

\[ Y(s) = K_2W(s) + K_1K_2V(s) \]

Also

\[ V(s) = X(s) + \frac{s}{s + \alpha} Y(s) \]

Therefore,

\[ Y(s) = K_2W(s) + K_1K_2 \left[ X(s) + \frac{s}{s + \alpha} Y(s) \right] \]

\[ Y(s) \left( 1 - \frac{K_1K_2s}{s + \alpha} \right) = K_2W(s) + K_1K_2X(s), \]

and

\[ Y(s) = \frac{K_2W(s) + K_1K_2X(s)}{1 - \frac{K_1K_2s}{s + \alpha}} \]

\[ = \frac{(s + \alpha)[K_2W(s) + K_1K_2X(s)]}{(1 - K_1K_2)s + \alpha} \]

#### S25.6

(a) The system function of the system given in Figure P25.6 must be determined first. So we write down the difference equation

\[ y[n] = x[n] + y[n - 1] + 4y[n - 2] \]

Taking the \( z\)-transform of the equation, we have

\[ Y(z)(1 - z^{-1} - 4z^{-2}) = X(z), \quad \text{or} \quad H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - z^{-1} - 4z^{-2}} \]

The poles of this system are located at

\[ z^2 - z - 4 = 0, \quad \text{or} \quad z = \frac{1}{2} \pm \frac{\sqrt{17}}{2} \]

Since \( |z| > 1 \) for at least one pole the system is unstable.

(b) With closed-loop feedback, the difference equation is


Thus,

\[ H(z) = \frac{z^2}{z^2 + (K - 1)z - 4} \]
The poles are now located at

\[ z = \frac{-(K - 1) \pm \sqrt{(K - 1)^2 + 16}}{2} \]

Note that the roots are purely real because the term inside the square root is always positive. For \( z = 1 \),

\[ 1 + \frac{K}{2} - \frac{1}{2} = \pm \frac{\sqrt{(K - 1)^2 + 16}}{2}, \]

\[ K + 1 = \pm \sqrt{(K - 1)^2 + 16} \]

Thus,

\[ K^2 + 2K + 1 = K^2 - 2K + 17, \]

\[ 4K = 16, \quad \text{or} \quad K = 4 \]

We can also calculate \( z_2 \):

\[ z_2 = -4 \]

Similarly, \( z_1 = -1, z_2 = 4 \) for \( K = -2 \). Observe the root locus in Figure S25.6-1.

![Figure S25.6-1](image)

Observe that if one of the poles is inside \( |z| = 1 \), the other is outside. Hence, the system is unstable for all values of \( K \).

(c) The difference equation can be written as

\[ y[n] = x[n] + y[n - 1] + (4 - K)y[n - 2] \]

Therefore,

\[ H(z) = \frac{z^2}{z^2 - z + (K - 4)} \]

In this case, the poles are located at

\[ z = \frac{1}{2} \pm \frac{\sqrt{17 - 4K}}{2} \]

For a stable system, we want

\[ |z| < 1, \]

\[ |z| = \left| \frac{1}{2} \pm \frac{\sqrt{17 - 4K}}{2} \right| \]

If we set \( 17 - 4K > 0 \), then

\[ \left| \frac{1}{2} \pm \frac{\sqrt{17 - 4K}}{2} \right| < 1, \]
or
\[
\pm \frac{\sqrt{17 - 4K}}{2} < \frac{1}{2},
\]
\[17 - 4K < 1, \quad K > 4\]

Now suppose \(17 - 4K < 0\). Then
\[
\left| \frac{1}{2} \pm j\sqrt{\frac{17 - 4K}{4}} \right| < 1 \quad \text{or} \quad \frac{17}{4} - K > -\frac{3}{4}, \quad -K > -\frac{5}{4}, \quad K < 5
\]

Thus, for \(K\) in the range \(4 < K < 5\), we have a stable system. The root locus is shown in Figure S25.6-2.

Figure S25.6-2

S25.7

(a) The dc gain of the amplifier is \(|H(0)| = |G|\).

(b) \(h(t) = Gae^{-at}u(t)\). Therefore, the time constant is \(1/a\).

(c) \(|H(j\omega_c)|^2 = \frac{G^2a^2}{a^2 + \omega_c^2} = \frac{1}{2}G^2\)

Thus \(\omega_c = \pm a\). Hence the bandwidth is \(a\).

(d) The closed-loop transfer function is
\[
V(s) = \frac{Ga}{s + a} = \frac{Ga}{1 + \frac{KG}{s + a} + s}
\]

From part (a), the time constant is
\[
\frac{1}{(1 + KG)a}
\]

From part (c), the bandwidth is \((1 + KG)a\). From part (a), the dc gain is
\[
\left| \frac{G}{1 + KG} \right|
\]

(e) We require \((GK + 1)a = 2a\). Hence, \(K = 1/G\). So the bandwidth becomes \(2a\).

The time constant is \(1/(2a)\), and \(|H(0)| = |G/2|\), the dc gain.
Resource: Signals and Systems
Professor Alan V. Oppenheim

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