21 Continuous-Time
Second-Order Systems

Solutions to
Recommended Problems

S21.1

(a) \( H_2(s) = \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st} \, dt = -\int_{-\infty}^{0} e^{-(a+s)t} \, dt \)

Following our previous arguments, we can integrate only if the function dies out as \( t \) goes to minus infinity. \( e^{-at} \) will die out as \( t \) goes to minus infinity only if \( Re(x) \) is negative. Thus we need \( Re(a + s) < 0 \) or \( Re(s) < -a \). For \( s \) in this range,

\[ H_2(s) = \frac{1}{a + s} \]

(b) (i) \( h_1(t) \) has a pole at \(-a\) and no zeros. Furthermore, since \( a > 0 \), the pole must be in the left half-plane. Since \( h_1(t) \) is causal, the ROC must be to the right of the rightmost pole, as given in D, Figure P21.1-4.

(ii) \( h_2(t) \) is left-sided; hence the ROC is to the left of the leftmost pole. Since \( a \) is positive, the pole is in the left half-plane, as shown in A, Figure P21.1-1.

(iii) \( h_3(t) \) is right-sided and has a pole in the right half-plane, as given in E, Figure P21.1-5.

(iv) \( h_4(t) \) is left-sided and has a pole in the right half-plane, as shown in C, Figure P21.1-3.

For a signal to be stable, its ROC must include the \( j\omega \) axis. Thus, C, D, and F qualify. B is an ROC that includes a pole, which is impossible; hence it corresponds to no signal.

S21.2

(a) By definition,

\[ X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} \, dt \]

We limit the integral to \((0, \infty)\) because of \( u(t) \), so

\[ X(s) = \int_{0}^{\infty} e^{-(1+s)t} \, dt = \frac{-1}{1 + s} \left. e^{-(1+s)t} \right|_{0}^{\infty} \]

If the real part of \((1 + s)\) is positive, i.e., \( Re[s] > -1 \), then

\[ \lim_{t \to \infty} e^{-(1+s)t} = 0 \]
Thus

\[ X(s) = \frac{0(-1)}{1 + s} - \frac{1(-1)}{1 + s} = \frac{1}{1 + s}, \quad Re[s] > -1 \]

The condition on \( Re[s] \) is the ROC and basically indicates the region for which \( 1/(1 + s) \) is equal to the integral defined originally. Similarly,

\[ H(s) = \int_{-\infty}^{\infty} e^{-2t} u(t) e^{-st} dt = \int_{0}^{\infty} e^{-(2+s)t} dt = \frac{1}{s + 2}, \quad Re[s] > -2 \]

(b) By the convolution property of the Laplace transform, \( Y(s) = H(s)X(s) \) in a manner similar to the property of the Fourier transform. Thus,

\[ Y(s) = \frac{1}{(s + 1)(s + 2)}, \quad Re[s] > -1, \]

where the ROC is the intersection of individual ROCs.

(c) Here we can use partial fractions:

\[
\frac{1}{(s + 1)(s + 2)} = \frac{A}{s + 1} + \frac{B}{s + 2},
\]

\[ A = Y(s)(s + 1) \bigg|_{s = -1} = 1, \]

\[ B = Y(s)(s + 2) \bigg|_{s = -2} = -1 \]

Thus,

\[ Y(s) = \frac{1}{s + 1} - \frac{1}{s + 2}, \quad Re[s] > -1 \]

Recognizing the individual Laplace transforms, we have

\[ y(t) = e^{-t}u(t) - e^{-2t}u(t) \]

(b) The property to be derived is

\[ x(t - t_0) \xrightarrow{\mathcal{L}} e^{-st_0}X(s), \]

with the same ROC as \( X(s) \).

Let \( y(t) = x(t - t_0) \). Then

\[ Y(s) = \int_{-\infty}^{\infty} y(t)e^{-st} dt = \int_{-\infty}^{\infty} x(t - t_0)e^{-st} dt \]

Let \( p = t - t_0 \). Then \( t = p + t_0 \) and \( dp = dt \). Substituting

\[ Y(s) = \int_{-\infty}^{\infty} x(p)e^{-sp}e^{-st_0} dp \]

Since we are not integrating over \( s \) or \( t_0 \), we can remove the \( e^{-st_0} \) term,

\[ Y(s) = e^{-st_0} \int_{-\infty}^{\infty} x(p)e^{-sp} dp = e^{-st_0}X(s) \]
Note that wherever $X(s)$ converges, the integral defining $Y(s)$ also converges; thus the ROC of $X(s)$ is the same as the ROC of $Y(s)$.

(b) Now we study one of the most useful properties of the Laplace transform. Let $x_1(t) \cdot x_2(t) \xrightarrow{L} X_1(s)X_2(s)$, with the ROC containing $R_1 \cap R_2$. Let

$$y(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t - \tau) \, d\tau$$

Then

$$Y(s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau)x_2(t - \tau)e^{-st} \, d\tau \, dt$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \int_{-\infty}^{\infty} x_2(t - \tau)e^{-st} \, dt \, d\tau$$

Suppose we are in a region of the $s$ plane where $X_2(s)$ converges. Then using the property shown in part (a), we have

$$\int_{-\infty}^{\infty} x_2(t - \tau)e^{-st} \, dt = e^{-st}X_2(s)$$

Substituting, we have

$$Y(s) = \int_{-\infty}^{\infty} x_1(\tau)e^{-st}X_2(s) \, d\tau = X_2(s) \int_{-\infty}^{\infty} x_1(\tau)e^{-st} \, d\tau$$

We can associate this last integral with $X_1(s)$ if we are also in the ROC of $x_1(t)$. Thus $Y(s) = X_2(s)X_1(s)$ for $s$ inside at least the region $R_1 \cap R_2$. It could happen that the ROC is larger, but it must contain $R_1 \cap R_2$.

(a) From the properties of the Laplace transform,

$$Y(s) = X(s)H(s)$$

A second relation occurs due to the differential equation. Since

$$\frac{d^2x(t)}{dt^2} \xrightarrow{L} s^2X(s)$$

and using the linearity property of the Laplace transform, we can take the Laplace transform of both sides of the differential equation, yielding

$$s^2Y(s) - sY(s) - 2Y(s) = X(s)$$

Therefore,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 - s - 2} = \frac{1}{(s - 2)(s + 1)}$$
The pole-zero plot is shown in Figure S21.4-1.

![Figure S21.4-1](image)

(b) (i) For a stable system, the ROC must include the $j\omega$ axis. Thus the ROC must be as drawn in Figure S21.4-2.

![Figure S21.4-2](image)

(ii) For a causal system, the ROC must be to the right of the rightmost pole, as shown in Figure S21.4-3.

![Figure S21.4-3](image)
(iii) For a system that is not causal or stable, we are left with an ROC that is to the left of $s = -1$, as shown in Figure S21.4-4.

![Figure S21.4-4](image)

(c) To take the inverse Laplace transform, we use the partial fraction expansion:

$$H(s) = \frac{1}{(s + 1)(s - 2)} = \frac{A}{s + 1} + \frac{B}{s - 2} = -\frac{1}{2} + \frac{1}{s - 2}$$

We now take the inverse Laplace transform of each term in the partial fraction expansion. Since the system is causal, we choose right-sided signals in both cases. Thus,

$$h(t) = -\frac{1}{2}e^{-t}u(t) + \frac{1}{4}e^{2t}u(t)$$

S21.5

$\omega = 0$: Since there is a zero at $s = 0$, $|H(j0)| = 0$. You may think that the phase is also zero, but if we move slightly on the $j\omega$ axis, $\angle H(j\omega)$ becomes

$$(\text{Angle to } s = 0) - (\text{Angle to } s = -1) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$\omega = 1$: The distance to $s = 0$ is 1 and the distance to $s = -1$ is $\sqrt{2}$. Thus

$$|H(j1)| = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

The phase is

$$(\text{Angle to } s = 0) - (\text{Angle to } s = -1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} = \angle H(j1)$$

$\omega = \infty$: The distance to $s = 0$ and $s = -1$ is infinite; however, the ratio tends to 1 as $\omega$ increases. Thus, $|H(j\infty)| = 1$. The phase is given by

$$\frac{\pi}{2} - \frac{\pi}{2} = 0$$
The magnitude and phase of \( H(j\omega) \) are given in Figure S21.5.

\[ |H(j\omega)| \]

\[ \angle H(j\omega) \]

Figure S21.5

The pole-zero plot is shown in Figure S21.6.

\[ \text{Im} \]

\[ s \text{ plane} \]

\[ \text{Re} \]

Because the zero at \( s = -5 \) is so far away from the \( j\omega \) axis, it will have virtually no effect on \( |H(j\omega)| \). Since there is a zero at \( \omega = 0 \) and poles near \( \omega = 2 \), we estimate a valley (actually a null) at \( \omega = 0 \) and a peak at \( \omega \approx \pm 2 \).
Solutions to Optional Problems

S21.7

(a) Let $y(t)$ be the system response to the excitation $x(t)$. Then the differential equation relating $y(t)$ to $x(t)$ is

$$\frac{d^2 y(t)}{dt^2} + 2\xi \omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$$

Integrating twice, we have

$$y(t) + 2\xi \omega_n \int_{-\infty}^{t} y(\tau) \, d\tau + \omega_n^2 \int_{-\infty}^{t} \int_{-\infty}^{\tau} y(\tau) \, d\tau \, d\tau' = \omega_n^2 \int_{-\infty}^{t} \int_{-\infty}^{\tau} x(\tau) \, d\tau \, d\tau'$$

or

$$y(t) = -2\xi \omega_n \int_{-\infty}^{t} y(\tau) \, d\tau - \omega_n^2 \int_{-\infty}^{t} \int_{-\infty}^{\tau} y(\tau) \, d\tau \, d\tau' + \omega_n^2 \int_{-\infty}^{t} \int_{-\infty}^{\tau} x(\tau) \, d\tau \, d\tau'$$

shown in Figure S21.7-1.
Recall that Figure S21.7-1 can be simplified as given in Figure S21.7-2.

(b) (i) For a constant $\omega_n$ and $0 \leq \zeta < 1$, $H(s)$ has a conjugate pole pair on a circle centered at the origin of radius $\omega_n$. As $\zeta$ changes from 0 to 1, the poles move from close to the $j\omega$ axis to $-\omega_n$, as shown in Figures S21.7-3, S21.7-4, and S21.7-5.

Figure S21.7-3 shows that for $\zeta \approx 0$ the pole is close to the $j\omega$ axis, so $|H(j\omega)|$ has a peak very near $\omega_n$. 

![Figure S21.7-2](image)

![Figure S21.7-3](image)
Figure S21.7-4 shows that the peaks are closer together and more spread out at $\zeta = 0.5$.

Figure S21.7-5 shows that at $\zeta \approx 1$ the poles are so close together and far from the $j\omega$ axis that $|H(j\omega)|$ has a single peak.
(ii) For constant $\xi$ between 0 and 1, the poles are located on two straight lines. As $\omega_n$ increases, the peak frequency increases as well as the bandwidth, as indicated in Figures S21.7-6 and S21.7-7.

Figure S21.7-6

Figure S21.7-7
(a) (i) The parallel implementation of $H(s)$, shown in Figure S21.8-1, can be drawn directly from the form for $H(s)$ given in the problem statement. The corresponding differential equations for each section are as follows:

\[
\begin{align*}
\frac{d^2y_1(t)}{dt^2} + \frac{dy_1(t)}{dt} + y_1(t) &= \frac{dx(t)}{dt}, \\
\frac{d^2y_2(t)}{dt^2} + 2\frac{dy_2(t)}{dt} + 2y_2(t) &= x(t), \\
y(t) &= y_1(t) + y_2(t)
\end{align*}
\]

\[\text{Figure S21.8-1}\]

(ii) To generate the cascade implementation, shown in Figure S21.8-2, we first express $H(s)$ as a product of second-order sections. Thus,

\[
H(s) = \frac{s(s^2 + 2s + 2) + (s^2 + s + 1)}{(s^2 + s + 1)(s^2 + 2s + 2)} = \frac{s^3 + 3s^2 + 3s + 1}{(s^2 + s + 1)(s^2 + 2s + 2)}
\]

Now we need to separate the numerator into two sections. In this case, the numerator equals $(s + 1)^3$, so an obvious choice is

\[(s + 1)(s^2 + 2s + 1)\]

Thus,

\[
H(s) = \left( \frac{s + 1}{s^2 + s + 1} \right) \left( \frac{s^2 + 2s + 1}{s^2 + 2s + 2} \right)
\]
The corresponding differential equations are as follows:

\[
\frac{d^2 r(t)}{dt^2} + \frac{dr(t)}{dt} + r(t) = x(t) + \frac{dx(t)}{dt},
\]

\[
\frac{d^2 y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = \frac{d^2 r(t)}{dt^2} + 2\frac{dr(t)}{dt} + r(t)
\]

(b) We see that we could have decomposed \( H(s) \) as

\[
H(s) = \left( \frac{s^2 + 2s + 1}{s^3 + s + 1} \right) \left( \frac{s + 1}{s^2 + 2s + 2} \right)
\]

Thus, the cascade implementation is not unique.

S21.9

(a) Decompose \( \sin \omega_0 t \) as

\[
\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}
\]

Then

\[
x_1(t) = \sin(\omega_0 t)u(t) = \frac{e^{j\omega_0 t}}{2j}u(t) - \frac{e^{-j\omega_0 t}}{2j}u(t)
\]

Using the transform pair given in the problem statement and the linearity property of the Laplace transform, we have

\[
X_1(s) = \frac{1}{2j} \left[ \frac{1}{s - j\omega_0} - \frac{1}{s + j\omega_0} \right]
\]

\[
= \frac{1}{2j} \frac{2j\omega_0}{s^2 + \omega_0^2} = \frac{\omega_0}{s^2 + \omega_0^2},
\]

with an ROC corresponding to \( \text{Re}(s) > 0 \).

(b) \( x_2(t) = e^{-z_1} \sin(\omega_0 t)u(t) \). Since

\[
e^{-z_1} \sin(\omega_0 t)u(t) \xrightarrow{L} X_1(s + 2),
\]
the ROC is shifted by 2. Therefore,
\[ e^{-2t} \sin(\omega_0 t)u(t) \leftarrow \mathcal{L} \frac{-\omega_0}{(s + 2)^2 + \omega_0^2} , \]
and the ROC is \( Re(s) > -2 \). Here we have used our answer to part (a).

(c) Since
\[ tx(t) \overset{\mathcal{L}}{\leftarrow} -\frac{dX(s)}{ds} , \]
with the same ROC as \( X(s) \), then
\[ te^{-2t}u(t) \overset{\mathcal{L}}{\leftarrow} -\frac{d}{ds} \left[ \frac{1}{(s + 2)} \right] , \]
thus
\[ te^{-2t}u(t) \overset{\mathcal{L}}{\leftarrow} \left[ \frac{-1}{(s + 2)^2} \right] = \frac{1}{(s + 2)^2} \]
with the ROC given by \( Re(s) > -2 \).

(d) Here we use partial fractions:
\[
\frac{s + 1}{(s + 2)(s + 3)} = \frac{A}{s + 2} + \frac{B}{s + 3},
\]
\[
A \left|_{s=-2} = \frac{s + 1}{s + 3} \right|_{s=-2} = -1, \quad B \left|_{s=-3} = \frac{s + 1}{s + 2} \right|_{s=-3} = \frac{-2}{-1} = 2,
\]
\[
\frac{s + 1}{(s + 2)(s + 3)} = \frac{-1}{s + 2} + \frac{2}{s + 3} \tag{S21.9-1}
\]
The ROC associated with the first term of eq. (S21.9-1) is \( Re(s) > -2 \) and the ROC associated with the second term is \( Re(s) > -3 \) to be consistent with the given total ROC. Thus,
\[ x(t) = -e^{-2t}u(t) + 2e^{-3t}u(t) \]
(e) From properties of the Laplace transform we know that
\[ x(t - T) \overset{\mathcal{L}}{\leftarrow} e^{-sT}X(s) , \]
with the same ROC as \( X(s) \). Since
\[ e^{-3t}u(t) \overset{\mathcal{L}}{\leftarrow} \frac{1}{s + 3} , \]
with an ROC given by \( Re(s) > -3, (1 - e^{-2s})/(s + 3) \) must correspond to
\[ x(t) = e^{-3t}u(t) - e^{-3t-2}u(t - 2) \]

S21.10

(a) (1), (2): An impulse has a constant Fourier transform whose magnitude is unaffected by a time shift. Hence, the Fourier transform magnitudes of (1) and (2) are shown in (c).

(3), (5): A decaying exponential corresponds to a lowpass filter; hence, (3) could be (a) or (d). By comparing it with (5), we see that (5) corresponds to \( kte^{-at}u(t) \), which has a double pole at \( -a \). Thus, (5) is a steeper lowpass filter than (3). Hence, (3) corresponds to (d) and (5) corresponds to (a).
(4), (7): These signals are of the form $e^{-at} \cos(\omega_0 t)u(t)$. For larger $a$, the poles are farther to the left. Hence $|H(j\omega)|$ for larger $a$ is less peaky. Thus, (4) corresponds to (f) and (7) corresponds to (g).

(6): If we convolve $x(t) = 1$ with $h(t)$ given in (6), we find that the output is zero. Thus (6) corresponds to a null at $\omega = 0$, either (b) or (h). Note that (6) can be thought of as an $h(t)$ given by (1) minus an $h(t)$ given by (3). Thus, the Fourier transform is the difference between a constant and a lowpass filter. Therefore, (6) is a highpass filter, or (b).

(b) (a), (d): These are simple lowpass filters that correspond to (i) or (ii). Since (a) is a steeper lowpass filter, we associate (a) with (ii) and (d) with (i).

(b), (h): These require a null at zero, and thus could correspond to (iii) or (viii). In the case of (iii), as $\omega$ increases, one pole-zero pair is canceled so that for large $\omega$, $H(s)$ looks like a lowpass filter. Hence, (b) corresponds to (viii) and (h) corresponds to (iii).

(c): Here we need a pole-zero plot that is an all-pass system. The only possible pole-zero plot is (vi).

(e): Here we need a null on the $j\omega$ axis, but not at $\omega = 0$. The only possibility is (v).

(f), (g): These are resonant second-order systems that could correspond to (iv) or (vii). Since poles closer to the $j\omega$ axis lead to peakier Fourier transforms, (f) must correspond to (iv) and (g) to (vii).
Resource: Signals and Systems  
Professor Alan V. Oppenheim  

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