19 Discrete-Time Sampling

Solutions to Recommended Problems

S19.1

$x[n]$ is given by

$$x[n] = (-1)^n = e^{jn\pi}$$

Hence, the Fourier transform of $x[n]$ is

$$X(\Omega) = \sum_{k=-\infty}^{\infty} \delta(\Omega - \pi - 2\pi k)$$

Now $p[n]$ can be written as

$$p[n] = \frac{1 + (-1)^n}{2}$$

Hence, its Fourier transform is given by

$$P(\Omega) = \frac{1}{2} \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k) + \frac{1}{2} \sum_{k=-\infty}^{\infty} \delta(\Omega - \pi - 2\pi k)$$

It is clear that $x_p[n] = p[n]$. Hence

$$X_p(\Omega) = P(\Omega)$$

S19.2

(a) $x_s[n]$ is $x[n]$ "stretched" by interspersing with zeros, as indicated in Figure S19.2-1.

Figure S19.2-1

$$X_s(\Omega) = \sum_{n=-\infty}^{\infty} x_s[n]e^{-jn\Omega}$$

$$= \sum_{n=-\infty}^{\infty} x_s[2n]e^{-j2n\Omega} + \sum_{n=-\infty}^{\infty} x_s[2n + 1]e^{-j(2n + 1)\Omega}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j(2\Omega)2n} + 0$$

$$= X(2\Omega)$$
(b) $x_d[n] = x[2n],$

$$X_d(\Omega) = \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[2n]e^{-j\Omega n}$$

$$= \frac{1}{2} \sum_{n=-\infty}^{\infty} (x[n] + (-1)^n x[n])e^{-j\Omega n/2}$$

$$= \frac{1}{2} \sum_{n=-\infty}^{\infty} x[n]e^{-j(\Omega/2)n} + \frac{1}{2} \sum_{n=-\infty}^{\infty} x[n]e^{-j(\Omega/2) - \pi n}$$

$$= \frac{1}{2} X(\frac{\Omega}{2}) + \frac{1}{2} X(\frac{\Omega}{2} - \pi)$$

(c) $X_d(\Omega) = X(2\Omega)$

$\frac{1}{2} X(\Omega/2)$ is indicated in Figure S19.2-3. Therefore, $X_d(\Omega)$ is as shown in Figure S19.2-4.
(a) For $N = 1$, $p[n] = 1$. Hence

$$P(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k),$$

as shown in Figure S19.3-1.

For $N = 2$,

$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n - 2k]$$

Hence

$$P(\Omega) = \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \pi k),$$

shown in Figure S19.3-2.

For $N = L$,

$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n - Lk]$$

Hence

$$P(\Omega) = \frac{2\pi}{L} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi k}{L}\right),$$

shown in Figure S19.3-3.
(b) $X_p(\Omega)$, the spectrum of $x_p[n]$, is proportional to the periodic convolution of $P(\Omega)$ and $X(\Omega)$. Consequently, with $P(\Omega)$ as indicated in Figure S19.3-3 and $X(\Omega)$ as indicated in Figure S19.3-4, $X_p(\Omega)$ is shown in Figure S19.3-5. In order that $x[n]$ be reconstructible from $x_p[n]$ using an ideal lowpass filter, aliasing must be avoided, which requires that

$$\Omega_M < \frac{2\pi}{N} - \Omega_M, \quad \text{or} \quad \Omega_M < \frac{\pi}{N}$$

\begin{center}
Figure S19.3-4
\end{center}

\begin{center}
Figure S19.3-5
\end{center}

(i) $\Omega_M = \frac{3\pi}{10}$. Therefore, to avoid aliasing,

$$\frac{\pi}{N} > \frac{3\pi}{10} \quad \text{or} \quad N < \frac{10}{3}$$

Since $N$ must be an integer, we require that $N \leq 3$. For $N = 3$, the cutoff frequency of the lowpass filter must be greater than $3\pi/10$ and less than

$$\frac{2\pi}{3} - \frac{3\pi}{10} = \left(\frac{11}{3}\right) \frac{\pi}{10}$$

(ii) $\Omega_M = \frac{3\pi}{5}$. To avoid aliasing,

$$\frac{\pi}{N} > \frac{3\pi}{5} \quad \text{or} \quad N < \frac{5}{3}$$

Since $N$ must be a positive integer, this requires that $N = 1$, i.e., $x[n]$ cannot be sampled.
S19.4

(a) The sampling period $T_1$ is 3 ms for the system in Figure P19.4-2 to be equivalent to the one in Figure P19.4-1.

(b) $X(\Omega)$ is sketched in Figure S19.4-1.

![Figure S19.4-1](image)

From the result of part (a), $Y(\Omega)$ is as shown in Figure S19.4-2.

![Figure S19.4-2](image)

S19.5

(a) Consider $x_d[n]$ and $x_{d3}[n]$, and let

$$x_{d3}[n] = x_d[n] + \alpha x_{d3}[n]$$

Then

$$x_{pd}[n] = \begin{cases} x_d[n/N] + \alpha x_{d3}[n/N], & n = 0, \pm N, \ldots, \\ 0, & \text{otherwise} \end{cases}$$

But

$$x_p[n] = \begin{cases} x_d[n/N], & n = 0, \pm N, \ldots, \\ 0, & \text{otherwise} \end{cases}$$

And

$$\alpha x_{pd}[n] = \begin{cases} \alpha x_{d3}[n/N], & n = 0, \pm N, \ldots, \\ 0, & \text{otherwise} \end{cases}$$
Hence,

\[ x_p[n] + ax_p[n] = \begin{cases} x_d[n/N] + ax_d[n/N], & n = 0, \pm N, \ldots, \\ 0, & \text{otherwise} \end{cases} \]

and

\[ x_p[n] = x_p[n] + ax_p[n] \]

So system A is linear.

(b) Take \( x_d[n] \) as shown in Figure S19.5-1, with \( N = 4 \).

Then \( x_p[n] \) is as shown in Figure S19.5-2.

Take \( x_d[n] = x_d[n + 1] \). Then \( x_p[n] \) is as shown in Figure S19.5-3.

Hence, system A is not time-invariant.

(c) \( x_p[n] = \begin{cases} x_d[n/N], & n = 0, \pm N, \ldots, \\ 0, & \text{otherwise} \end{cases} \)

Hence

\[ X_p(\Omega) = \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\Omega(nN)} = X_d(N\Omega), \]

as shown in Figure S19.5-4.
(d) $X(\Omega)$ is as shown in Figure S19.5-5 for exact bandlimited interpolation.

(a) $x_p(t)$ is sketched in Figure S19.6-1, and $Y_p(\omega)$ is sketched in Figure S19.6-2.
(b) $X_p(\omega)$ is sketched in Figure S19.6-3, and $y_p(t)$ is sketched in Figure S19.6-4.

(c) Yes, $y_p(t)$ is periodic and this is reflected in $Y_p(\omega)$, which contains impulses.

**Solutions to Optional Problems**

S19.7

(a) $x_p[n] = \begin{cases} x[n], & n = 0, \pm 2, \pm 4, \ldots, \\ 0, & n = \pm 1, \pm 3, \pm 5, \ldots \end{cases}$

This is sketched in Figure S19.7-1.

Similarly, $x_d[n] = x[2n]$, as shown in Figure S19.7-2.
(b) \(X_p(\Omega)\) is obtained as follows:

\[
x_p[n] = \frac{1}{2}x[n] + \frac{1}{2}(-1)^nx[n]
\]

\[
X_p(\Omega) = \frac{1}{2}X(\Omega) + \frac{1}{2}X(\Omega - \pi)
\]

and

\[
X_d(\Omega) = \frac{1}{2}X\left(\frac{\Omega}{2}\right) + \frac{1}{2}X\left(\frac{\Omega}{2} - \pi\right),
\]

which are shown in Figures S19.7-3 and S19.7-4. See Problem P19.2(b).
Aliasing will be just avoided when the sampled spectra will look as shown in Figure S19.8-1.

Hence, we require that

\[
\frac{2\pi}{N} > \frac{6\pi}{11}, \quad \text{or} \quad N < \frac{22}{6} \Rightarrow N \leq 3
\]

Consequently, aliasing is avoided if \(1 \leq N \leq 3\). \(X_p(\Omega)\) for \(N = 1, 2, \) and \(3\) are shown in Figure S19.8-2.
Discrete-Time Sampling / Solutions
S19-11

(b) An appropriate $H(\Omega)$ is shown in Figure S19.8-3.

![Figure S19.8-3](image)

S19.9

(a) $y[n] = \frac{x[3n] + x[3n + 1] + x[3n + 2]}{3}$

- $y[-4] = 0$
- $y[-3] = 0$
- $y[-2] = \frac{1}{3}x[-4] = \frac{1}{3}$
- $y[-1] = \frac{x[-3] + x[-2] + x[-1]}{3} = \frac{2}{3}$
- $y[0] = \frac{x[0] + x[1] + x[2]}{3} = \frac{4}{3}$
- $y[1] = \frac{x[3] + x[4] + x[5]}{3} = \frac{1}{3}$
- $y[2] = 0$
- $y[3] = 0$

Hence, $y[n]$ can be sketched as in Figure S19.9.

![Figure S19.9](image)

(b) If

$z[n] = \frac{1}{3}[x[n] + x[n + 1] + x[n + 2]]$, for all $n$
and

\[ y[n] = z[3n], \]

we have expressed the processing as a combination of filtering and decimation.

**S19.10**

If \( h[0] = 1 \) and \( h[n] = 0 \) for \( n = kN, k \neq 0 \), it is easy to see that the samples \( x_0[n] \) that came from \( x[n] \) will be unaffected. Hence,

\[ y[kN] = x[k], \quad \text{for all } k \]