17 Interpolation

Solutions to Recommended Problems

S17.1

It is more convenient to solve this problem in the time domain than in the frequency domain. Since \( x_p(t) = x(t)p(t) \) and \( p(t) \) is an impulse train, \( x_p(t) \) is a sampled version of \( x(t) \), as shown in Figure S17.1-1.

\[
x_p(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)
\]

\[
= \sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT)
\]

Since \( y(t) = x_p(t) * h(t) \) and \( x_p(t) \) is impulsive, the convolution carried out in the time domain is as shown in Figure S17.1-2.

Here we have that \( x(t) \) is sampled by a rectangular pulse train as opposed to an impulse train.
Since \( h(t) \ast (1/T)h(t) \), shown in Figure S17.1-3, is wider than the sampling period \( T \), the resultant \( w(t) \) is not a triangularly sampled version of \( x(t) \). \( w(t) \) consists of the superposition of waveforms shown in Figure S17.1-4.

We note that this superposition is actually a linear interpolation between the samples of \( x(t) \). For example, Figure S17.1-5 convolved with Figure S17.1-6 equals Figure S17.1-7.
Now adding the shifted and scaled triangles yields Figure S17.1-8, which we see is the linear interpolation between samples of $x_n(t)$.

Now since $(1/T)h(t) \ast h(t)$ is Figure S17.1-9, we expect that $w(t)$ is the linear interpolation of $x_n(t)$ shifted right by $T$, shown in Figure S17.1-10.
$y(t)$ in all cases is the superposition of two signals.

(a) $y_1(t)$

(b) $y_2(t)$

(c) $y_3(t)$

Figure S17.2-1

Figure S17.2-2

Figure S17.2-3
We have

\[
\frac{2\pi}{T} - \omega_m > \omega_m,
\]

\[
\frac{2\pi}{T} > 2\omega_m,
\]

\[
\omega_s > 2\omega_m,
\]

where \( \omega_s \) is the sampling frequency. To ensure no aliasing we require that \( T < \pi/\omega_m \).

**b)** To recover \( x(t) \) from \( x_p(t) \), we must interpolate. We have previously shown that by lowpass filtering, the spectrum is recovered, assuming that sampling has been performed at a sufficiently high rate. The interpolation may be done in many different ways, however, depending on the cutoff frequency we choose for the lowpass filter. For example, any of the filters \( H_1(\omega) \), \( H_2(\omega) \), and \( H_3(\omega) \) in Figures S17.3-3 to S17.3-5 may be used to interpolate \( x_p(t) \), whose Fourier transform is shown in Figure S17.3-2, to yield \( x(t) \).
(e) If

\[ x_p(t) = \sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT) \]

and \( H(\omega) \) is an ideal lowpass filter whose Fourier transform is shown in Figure S17.3-6, then

\[ h(t) = \frac{T \sin \omega_c t}{\pi t} = \frac{T \omega_c}{\pi} \text{sinc} \frac{\omega_c t}{\pi} \]

and

\[ x(t) = x_p(t) \ast h(t) = \frac{T \omega_c}{\pi} \sum_{n=-\infty}^{\infty} x(nT) \text{sinc} \left( \frac{t - nT}{\pi} \right) \]

![Figure S17.3-6](image)

**Figure S17.3-6**

**S17.4**

We want to choose \( H(\omega) \) so that the cascade of the two filters is an ideal lowpass filter. In this example,

\[ G(\omega) = 2 \sin \frac{\omega \Delta/2}{\omega} e^{-j\omega \Delta/2} \]

so that

\[ H(\omega) = \frac{\omega}{2 \sin \omega \Delta/2} e^{j\omega \Delta/2}, \quad |\omega| < \pi/\Delta, \]

\[ H(\omega) = 0 \quad \text{otherwise} \]
Figure S17.5

Solutions to Optional Problems

S17.6

(a) We want $h_d[n]$ such that $x_p[n] * h_d[n] = x_d[n]$. We sketch $h_d[n]$ in Figure S17.6-1.
(b) \( x_d[n] = x[n] \ast h_d[n] \)

If there is no aliasing, we can recover \( x[n] \) from \( x_d[n] \) by proper filtering. Since

\[
H_d(\Omega) = \frac{1 - e^{-j\pi N}}{1 - e^{-j\Omega}} = e^{-j\pi(N-1)/2} \frac{\sin \Omega N/2}{\sin \Omega/2},
\]

we require

\[
H(\Omega) = \begin{cases} 
  N/H_d(\Omega), & |\Omega| \leq \frac{\pi}{N}, \\
  0, & \frac{\pi}{N} < |\Omega| \leq \pi
\end{cases}
\]

\[
= \begin{cases} 
  Ne^{j\pi(N-1)/2} \frac{\sin \Omega/2}{\sin \Omega N/2}, & |\Omega| \leq \frac{\pi}{N} \\
  0, & \frac{\pi}{N} < |\Omega| \leq \pi
\end{cases}
\]

(c) \( h_1[n] = \frac{1}{N} (h_d[n] \ast h_d[-n]) \), so \( h_d[n] \) is a triangular discrete-time pulse, as shown in Figure S17.6-2.

![Figure S17.6-2](attachment:image.png)

(d) We require that

\[
H(\Omega) = \begin{cases} 
  \frac{N}{H_1(\Omega)}, & \text{for } |\Omega| \leq \frac{\pi}{N}, \\
  0, & \text{for } \frac{\pi}{N} < |\Omega| < \pi
\end{cases}
\]

From part (c),

\[
H(\Omega) = N \left[ \frac{N}{|H_0(\Omega)|^2} \right] = \begin{cases} 
  N^2 \left( \frac{\sin \Omega/2}{\sin \Omega N/2} \right)^2, & |\Omega| \leq \frac{\pi}{N}, \\
  0, & \frac{\pi}{N} < |\Omega| < \pi
\end{cases}
\]

S17.7

(a) Taking Fourier transforms of both sides of the LCCDE yields

\[
-j\omega Y_s(\omega) + Y_s(\omega) = 1,
\]

\[
Y_s(\omega) = \frac{1}{1 + j\omega},
\]
so \( y_c(t) = e^{-t}u(t) \). By examining the sampler followed by conversion to an impulse train, we note that

\[
y[n] = y_c(nT) = e^{-nT}u[n]
\]

(b) Since \( y[n] = e^{-nT}u[n] \), we can take the Fourier transform to yield

\[
Y(\Omega) = \sum_{n=0}^{\infty} e^{-nT}e^{-j\Omega n} = \frac{1}{1 - e^{(-T+j\Omega)}}
\]

In order for \( w[n] = \delta[n] \), we require \( W(\Omega) = 1 \) for all \( \Omega \). Thus,

\[
Y(\Omega)H(\Omega) = W(\Omega) = 1,
\]

\[
H(\Omega) = 1 - e^{(-T+j\Omega)}
\]

Now since \( H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n} \), we see by inspection that

\[
h[0] = 1,
\]
\[
h[1] = -e^{-T},
\]
\[
h[n] = 0, \quad n \neq 0, 1
\]

S17.8

(a) Since \( X_P(\omega) = X(\omega)P(\omega) \), we conclude that \( X_P(\omega) \) is as shown in Figure S17.8-1.

(b) From the convolution theorem,

\[
y(t) = x(t) * p(t)
\]

and

\[
p(t) = \frac{1}{\omega_s} \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{2\pi n}{\omega_s}\right)
\]
The fact that
\[
\mathcal{F}(p(t)) = \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)
\]
can be easily verified. Therefore,
\[
y(t) = x(t) * \frac{1}{\omega_s} \sum_{n=-\infty}^{\infty} \delta \left( t - \frac{2\pi n}{\omega_s} \right)
\]
(c) We see from the sketch in Figure S17.8-2 that for no time-domain aliasing,
\[
T \leq \frac{2\pi}{\omega_s} - T, \quad \text{so} \quad \omega_s \leq \frac{\pi}{T}
\]

(d) \(x(t)\) may be recovered from \(y(t)\), assuming that no time-domain aliasing has occurred, by low-time filtering \(y(t)\) from \(t = -T\) to \(T\) and applying a gain of \(\omega_s\).
Resource: Signals and Systems
Professor Alan V. Oppenheim

The following may not correspond to a particular course on MIT OpenCourseWare, but has been provided by the author as an individual learning resource.

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.