13 Continuous-Time Modulation

Solutions to Recommended Problems

S13.1

(a) By the shifting property,

\[ x(t)e^{j3\omega_c t} \rightarrow X(\omega - 3\omega_c) = Y(\omega) \]

The magnitude and phase of \( Y(\omega) \) are given in Figure S13.1-1.

(b) Since \( e^{j3\omega_c t + j\pi/2} = e^{j\pi/2}e^{j3\omega_c t} \), we are modulating the same carrier as in part (a) except that we multiply the result by \( e^{j\pi/2} \). Thus

\[ Y(\omega) = e^{j\pi/2}X(\omega - 3\omega_c) \]

Note in Figure S13.1-2 that the magnitude of \( Y(\omega) \) is unaffected and that the phase is shifted by \( \pi/2 \).

(c) Since

\[ \cos 3\omega_c t = \frac{e^{j3\omega_c t}}{2} + \frac{e^{-j3\omega_c t}}{2}, \]

\[ \sin 3\omega_c t = \frac{e^{j3\omega_c t} - e^{-j3\omega_c t}}{2j}, \]

\[ \tan 3\omega_c t = \frac{e^{j3\omega_c t} - e^{-j3\omega_c t}}{e^{j3\omega_c t} + e^{-j3\omega_c t}}. \]
we can think of modulation by \( \cos 3\omega_0 t \) as the sum of modulation by

\[
\frac{e^{j3\omega_0 t}}{2} \quad \text{and} \quad \frac{e^{-j3\omega_0 t}}{2}
\]

Thus, the magnitude and phase of \( Y(\omega) \) are as shown in Figure S13.1-3. Note the scaling in the magnitude.

![Figure S13.1-3](image)

(d) We can think of modulation by \( \sin 3\omega_0 t \) as the sum of modulation by

\[
\frac{e^{j3\omega_0 t} - j/2}{2} \quad \text{and} \quad \frac{e^{-j3\omega_0 t} - j/2}{2}
\]

Thus, the magnitude and phase of \( Y(\omega) \) are as given in Figure S13.1-4. Note the scaling by \( \frac{1}{2} \) in the magnitude.

![Figure S13.1-4](image)

(e) Since the phase terms are different in parts (c) and (d), we cannot just add spectra. We need to convert \( \cos 3\omega_0 t + \sin 3\omega_0 t \) into the form \( A \cos(3\omega_0 t + \theta) \). Note
that

\[
\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta
\]

Let \( \alpha = 3\omega t \) and \( \beta = \pi/4 \). Then

\[
\cos \left( 3\omega t - \frac{\pi}{4} \right) = \frac{1}{\sqrt{2}} (\cos 3\omega t + \sin 3\omega t)
\]

Thus

\[
\cos 3\omega t + \sin 3\omega t = \sqrt{2} \cos \left( 3\omega t - \frac{\pi}{4} \right)
\]

Now we write \( c(t) \) as

\[
\frac{\sqrt{2}}{2} e^{j\beta \omega t - (\pi/4)} + \frac{\sqrt{2}}{2} e^{-j\beta \omega t - (\pi/4)}
\]

Modulating by each exponential separately and then adding yields the magnitude and phase given in Figure S13.1-5. (Note the scaling in the magnitude.)

\[\text{Figure S13.1-5}\]

**S13.2**

In Figure S13.2-1 we redraw the system with some auxiliary signals labeled.
By the modulation property, $R_1(\omega)$, the Fourier transform of $r_1(t)$, is

$$R_1(\omega) = \frac{1}{2\pi} [X(\omega) \ast S(\omega)]$$

Since $S(\omega)$ is composed of impulses, $R_1(\omega)$ is a repetition of $X(\omega)$ centered at $-2\omega_c$, 0, and $2\omega_c$, and scaled by $1/(2\pi)$. See Figure S13.2-2.

(a) Since $m(t) = d(t) = 1$, $y(t)$ is $r_1(t)$ filtered twice by the same ideal lowpass filter with cutoff at $\omega_c$. Thus, comparing the resulting Fourier transform of $y(t)$, shown in Figure S13.2-3, we see that $y(t) = 1/(2\pi)x(t)$, which is nonzero.

(b) Modulating $r_1(t)$ by $e^{j\omega_c t}$ yields $R_1(\omega - \omega_c)$ as shown in Figure S13.2-4.
Similarly, modulating by $e^{-j\omega t}$ yields $R_i(\omega + \omega_c)$ as shown in Figure S13.2-5.

![Figure S13.2-5](image)

Since $\cos \omega t = (e^{j\omega t} + e^{-j\omega t})/2$, modulating $r_i(t)$ by $\cos \omega t$ yields a Fourier transform of $r_2(t)$ given by

$$R_2(\omega) = \frac{R_i(\omega - \omega_c) + R_i(\omega + \omega_c)}{2}$$

Thus, $R_2(\omega)$ is as given in Figure S13.2-6.

![Figure S13.2-6](image)

After filtering, $R_3(\omega)$ is given as in Figure S13.2-7.

![Figure S13.2-7](image)
$R_4(\omega)$ is given by shifting $R_3(\omega)$ up and down by $\omega_c$ and dividing by 2. See Figure S13.2-8.

![Figure S13.2-8](image)

After filtering, $Y(\omega)$ is as shown in Figure S13.2-9.

![Figure S13.2-9](image)

Comparing $Y(\omega)$ and $X(\omega)$ yields

$$y(t) = \frac{1}{4\pi} x(t)$$

(e) Since

$$\sin \omega_c t = \frac{\text{e}^{j\omega_c t} - \text{e}^{-j\omega_c t}}{2j},$$

then

$$R_4(\omega) = \frac{R_3(\omega - \omega_c) - R_3(\omega + \omega_c)}{2j},$$

which is drawn in Figure S13.2-10.

![Figure S13.2-10](image)

After filtering, $R_3(\omega) = 0$. Therefore, $y(t) = 0$. 
(d) In this case, it is not necessary to know \( r_3(t) \) exactly. Suppose \( r_3(t) \) is nonzero, with \( R_3(\omega) \) given as in Figure S13.2-11.

After modulating by \( d(t) = \cos 2\omega t \), \( R_d(\omega) \) is given as in Figure S13.2-12.

After filtering, \( y(t) = 0 \) since \( R_d(\omega) \) has no energy from \(-\omega_c\) to \(\omega_c\).

(e) For this part, let us calculate \( R_2(\omega) \) explicitly.

\[
R_2(\omega) = \frac{R_1(\omega - 2\omega_c) + R_1(\omega + 2\omega_c)}{2},
\]

which is drawn in Figure S13.2-13.
After filtering, $R_3(\omega)$ is as shown in Figure S13.2-14.

![Figure S13.2-14](image)

Modulating again yields $R_4(\omega)$ as shown in Figure S13.2-15.

![Figure S13.2-15](image)

Finally, filtering $R_4(\omega)$ gives the Fourier transform of $y(t)$, shown in Figure S13.2-16.

![Figure S13.2-16](image)

Thus,

$$y(t) = \frac{1}{8\pi} x(t)$$

**S13.3**

(a) The demodulator signal $w(t)$ is related to $x(t)$ via

$$w(t) = (\cos \omega_st)(\cos \omega_r t x(t)$$
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S13-9

Since \( \cos A \cos B = \frac{1}{2} \{ \cos(A - B) + \cos(A + B) \} \),
\[
u(t) = \frac{1}{2} \{ \cos(\Delta \omega) t + \cos(\Delta \omega + 2\omega_c) t \} x(t)
= \frac{1}{2} \{ \cos(\Delta \omega) t \} x(t) + \frac{1}{2} \{ \cos(\Delta \omega + 2\omega_c) t \} x(t)
\]
The first term is bandlimited to \( \pm (\omega_M + \omega_c) \), while the second term is bandlimited from \( \Delta \omega + 2\omega_c - \omega_M \) to \( \Delta \omega + 2\omega_c + \omega_M \). Thus after filtering, only the first term remains. Therefore, the output of the demodulator lowpass filter is given by \( \frac{1}{2} x(t) \cos \Delta \omega t \).

(b) Consider first \( |\Delta \omega| > \omega_M \). Then for \( X(\omega) \) as given, \( \frac{1}{2} x(t) \cos \Delta \omega t \) has a Fourier transform as shown in Figure S13.3-1.

\[\text{Figure S13.3-1}\]

For \( |\Delta \omega| < \omega_M \), there is some overlap. See Figure S13.3-2.

\[\text{Figure S13.3-2}\]

S13.4

(a) In this case,
\[
y(t) = [A + \cos \omega_M t] \cos(\omega_c t + \theta_c)
\]
But
\[
\cos \omega_M t \cos(\omega_c t + \theta_c) = \frac{1}{2} \{ \cos((\omega_M - \omega_c) t - \theta_c) + \cos((\omega_M + \omega_c) t + \theta_c) \}
\]
Thus,
\[
y(t) = A \cos(\omega_c t + \theta_c) + \frac{1}{2} \cos((\omega_M - \omega_c) t - \theta_c) + \frac{1}{2} \cos((\omega_M + \omega_c) t + \theta_c)
= \frac{A e^{j\theta_c}}{2} e^{j\omega_c t} + \frac{A e^{-j\theta_c}}{2} e^{-j\omega_c t} + \frac{1}{4} e^{-j\omega_M \omega_c} e^{j(\omega_M + \omega_c) t}
+ \frac{1}{4} e^{j\omega_M \omega_c} e^{-j(\omega_M + \omega_c) t} + \frac{1}{4} e^{-j\omega_M \omega_c} e^{j(\omega_M + \omega_c) t}
\]
We recognize that the preceding expression is a Fourier series expansion. Using Parseval's theorem for the Fourier series, we have

\[
\frac{1}{T_0} \int_{t_0}^{t_0} |y(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |a_k|^2 = P_y
\]

Thus,

\[
P_y = 2 \left( \frac{A}{2} \right)^2 + 4 \left( \frac{1}{4} \right)^2 = \frac{A^2}{2} + \frac{1}{4}
\]

Since

\[
m = \frac{\max |x(t)|}{A} = \frac{1}{A},
\]

then

\[
P_y = \frac{1}{2m^2} + \frac{1}{4},
\]

as shown in Figure S13.4-1.

(b) The power in the sidebands is found from \(P_y\) when \(A = 0\). Thus, \(P_y = \frac{1}{4}\) and the efficiency is

\[
E = \frac{\frac{1}{4}}{1/(2m^2) + \frac{1}{4}} = \frac{m^2}{2 + m^2},
\]

which is sketched in Figure S13.4-2.
Solutions to Optional Problems

S13.5

(a) Using the identity for \( \cos(A + B) \), we have

\[
A(t) \cos(\omega_c t + \theta_c) = A(t) \left( \cos \theta_c \cos \omega_c t - \sin \theta_c \sin \omega_c t \right)
\]

Thus, we see that

\[
x(t) = A(t) \cos \theta_c,
\]

\[
y(t) = -A(t) \sin \theta_c.
\]

Therefore,

\[
z(t) = A(t) \cos(\omega_c t + \theta_c)
\]

\[
= x(t) \cos \omega_c t + y(t) \sin \omega_c t.
\]

(b) Consider modulating \( z(t) \) by \( \cos \omega_c t \). Then

\[
z(t) \cos \omega_c t = x(t) \cos^2 \omega_c t + y(t) \sin \omega_c t \cos \omega_c t.
\]

Using trigonometric identities, we have

\[
z(t) \cos \omega_c t = \frac{x(t)}{2} + \frac{x(t)}{2} \cos 2\omega_c t + \frac{y(t)}{2} \sin 2\omega_c t.
\]

If we use an ideal lowpass filter with cutoff \( \omega_c \), and if \( A(t) \), and thus \( x(t) \), is bandlimited to \( \pm \omega_c \), then we recover the term \( x(t)/2 \). Thus the processing is as shown in Figure S13.5-1.

(c) Similarly, consider

\[
z(t) \sin \omega_c t = x(t) \cos \omega_c t \sin \omega_c t + y(t) \sin^2 \omega_c t
\]

\[
= \frac{x(t)}{2} \sin 2\omega_c t + \frac{y(t)}{2} - \frac{y(t)}{2} \cos 2\omega_c t.
\]

Filtering \( z(t) \sin \omega_c t \) with the same filter as in part (b) yields \( y(t) \), as shown in Figure S13.5-2.
(d) We can readily see that

\[ x(t)^2 + y(t)^2 = A(t)^2 (\cos^2 \theta_c + \sin^2 \theta_c) = A(t)^2 \]

Therefore, \( A(t) = \sqrt{x(t)^2 + y(t)^2} \). The block diagram in Figure S13.5-3 summarizes how to recover \( A(t) \) from \( z(t) \).

Note that to be able to recover \( A(t) \) in this way, the Fourier transform of \( A(t) \) must be zero for \( \omega > |\omega_c| \) and \( A(t) > 0 \). Also note that we are implicitly assuming that \( A(t) \) is a real signal.

**S13.6**

From Figures P13.6-1 to P13.6-3, we can relate the Fourier transforms of all the signals concerned.

\[ S_1(\omega) = \frac{1}{2j} \left[ X\left(\omega - \frac{\omega_0}{2}\right) - X\left(\omega + \frac{\omega_0}{2}\right) \right] \]

\[ S_2(\omega) = \frac{1}{2} \left[ X\left(\omega - \frac{\omega_0}{2}\right) + X\left(\omega + \frac{\omega_0}{2}\right) \right] \]

Thus, \( S_1(\omega) \) and \( S_2(\omega) \) appear as in Figure S13.6-1.
After filtering, $S_3(\omega)$ and $S_4(\omega)$ are given as in Figure S13.6-2.

$S_5(\omega)$ is as follows (see Figure S13.6-3):

$$S_5(\omega) = \frac{1}{2j} \left[ S_3 \left( \omega - \frac{\omega_0}{2} \right) - S_3 \left( \omega + \frac{\omega_0}{2} \right) \right]$$

Note that the amplitude is reversed since $(1/2j)(1/2j) = -\frac{1}{4}$. 
\( S_6(\omega) \) is as follows and as shown in Figure S13.6-4.

\[
S_6(\omega) = \frac{1}{2} \left[ S_4(\omega - \omega_c - \frac{\omega_0}{2}) + S_4(\omega + \omega_c + \frac{\omega_0}{2}) \right]
\]

Finally, \( Y(\omega) = S_5(\omega) + S_6(\omega) \), as shown in Figure S13.6-5.

Thus, \( y(t) \) is a single-sideband modulation of \( x(t) \).

**S13.7**

Note that

\[
q_i(t) = [s_1(t)\cos \omega_0 t + s_2(t)\sin \omega_0 t] \cos \omega_0 t = s_1(t)\cos^2 \omega_0 t + s_2(t)\sin \omega_0 t \cos \omega_0 t
\]

Using trigonometric identities, we have

\[
q_i(t) = \frac{1}{2}s_1(t) + \frac{1}{2}s_2(t)\cos 2\omega_0 t + \frac{1}{2}s_2(t)\sin 2\omega_0 t
\]

Thus, if \( s_i(t) \) is bandlimited to \( \pm \omega_0 \) and we use the filter \( H(\omega) \) as given in Figure S13.7, \( y_i(t) \) will then equal \( s_i(t) \).
Similarly,
\[ q_\omega(t) = s_1(t) \cos \omega_0 t \sin \omega_0 t + s_2(t) \sin^2 \omega_0 t \]
\[ = \frac{s_1(t)}{2} \sin 2\omega_0 t + \frac{s_2(t)}{2} - \frac{s_2(t)}{2} \cos 2\omega_0 t \]

Using the same filter and imposing the same restrictions on \( s_\omega(t) \), we obtain \( y_\omega(t) = s_\omega(t) \).

**S13.8**

(a) \( X(\omega) \) is given as in Figure S13.8-1.

For \( Y(\omega) \), the spectrum of the scrambled signal is as shown in Figure S13.8-2.

Thus, \( X(\omega) \) is reversed for \( \omega > 0 \) and \( \omega < 0 \).
(b) Suppose we multiply \( x(t) \) by \( \cos \omega_M t \). Denoting \( z(t) = x(t)\cos \omega_M t \), we find that \( Z(\omega) \) is composed of scaled versions of \( X(\omega) \) centered at \( \pm \omega_M \). See Figure S13.8-3.

![Figure S13.8-3](image)

Filtering \( z(t) \) with an ideal lowpass filter with a gain of 2 yields \( y(t) \), as shown in Figure S13.8-4.

![Figure S13.8-4](image)

(c) Suppose we use the same system to recover \( x(t) \). Let \( y(t)\cos \omega_M t = r(t) \). Then \( R(\omega) \) is as given in Figure S13.8-5.

![Figure S13.8-5](image)

Filtering with the same lowpass filter yields \( x(t) \).