(a) The impulse response is real because
\[ h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)e^{j\omega t} d\omega, \]
\[ h^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H^*(\omega)e^{-j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)e^{-j\omega t} d\omega \]
where we used the fact that \( H(\omega) = H^*(\omega) = H(-\omega). \)
The impulse response is even because
\[ h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)e^{j\omega t} d\omega, \]
\[ h(-t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)e^{-j\omega t} d\omega \]
Since \( H(-\omega) = H(\omega), \)
\[ h(-t) = h(t). \]
The impulse response is noncausal because \( h(-t) \neq h(t). \)

(b) \( x(t) = \sum_{n=-\infty}^{\infty} \delta(t - 9n), \)
\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi kt)/T}, \]
\[ a_k = \frac{1}{T} \int_{0}^{T} x(t)e^{-j(2\pi kt)/T} dt \]
Here \( T = 9, \) so
\[ a_k = \frac{1}{9} \quad \text{and} \quad \mathcal{F}\{e^{j(2\pi kt)/T}\} = 2\pi\delta \left( \omega - \frac{2\pi k}{T} \right) \]
Consequently, the Fourier transform of the filter input is as shown in Figure S12.1-1.
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S12-2

Since $Y(\omega) = H(\omega)X(\omega)$, the Fourier transform of the filter output is as shown in Figure S12.1-2.

![Figure S12.1-2](image.png)

(c) We determine $y(t)$ by performing an inverse Fourier transform on $Y(\omega)$ as found in part (b). Using superposition, we have

\[
y(t) = \frac{1}{9} + \frac{2}{9} \cos\left(\frac{2\pi t}{9}\right)
\]

S12.2

From the filter frequency response plots we can determine that

\[
H(\omega) = 0.25e^{-j(\pi/8)} \quad \text{at } \omega = \omega_1 = \pi,
\]

\[
H(\omega) = 0.5e^{-j(\pi/4)} \quad \text{at } \omega = \omega_2 = 2\pi
\]

Using superposition, we easily determine $y(t)$ to be

\[
y(t) = 0.25 \sin(\pi t + \pi/8) + \cos\left(2\pi t - \frac{7\pi}{12}\right)
\]

S12.3

(a) $RC \frac{dv_c}{dt} + v_c = v_i$

Taking the Fourier transform of this equation, we have

\[
(RCj\omega + 1)V_c(\omega) = V_i(\omega)
\]

We now define

\[
H_i(\omega) = \frac{V_c(\omega)}{V_i(\omega)} = \frac{1}{1 + j\omega RC}
\]

We can see from this expression that $v_c(t)$ is a lowpass version of $v_i(t)$. 
The magnitude and phase of $H_i(\omega)$ are given in Figure S12.3-1.

(b) \[
RC \frac{d(v_s - v_r)}{dt} + v_s - v_r = v_i,
\]
\[
RCj\omega V_i(\omega) - RCj\omega V_s(\omega) - V_r(\omega) = 0,
\]
\[
(j\omega RC)V_s(\omega) = (1 + j\omega RC)V_r(\omega),
\]
\[
H_2(\omega) = \frac{V_s(\omega)}{V_i(\omega)} = \frac{j\omega RC}{1 + j\omega RC}
\]
The magnitude and phase of $H_2(\omega)$ are given in Figure S12.3-2.

Figure S12.3-2

(c) The cutoff frequencies are $\omega_c = 1/RC$ in both cases.

(d) $\frac{V(\omega)}{V_c(\omega)} = 1 - H_1(\omega) = \frac{j\omega RC}{1 + j\omega RC} = H_2(\omega)$

This is the same frequency response as sketched in part (b). We have transformed a lowpass into a highpass filter by a feed-forward system. The cutoff frequency, as in part (c), is $\omega_c = 1/RC$.

S12.4

Consider $0 \leq \Omega_0 \leq \pi$. In this range, the gain of the filter $|H(\Omega)|$ is $\Omega_0$. The phase shift for the positive frequency component is $+ \pi/2$ and the shift for the negative frequency component is $- \pi/2$. Since

$$x[n] = \cos (\Omega_0 n + \theta) = \frac{1}{2}[e^{j(\Omega_0 n + \theta)} + e^{-j(\Omega_0 n + \theta)}],$$

$$y[n] = \frac{\Omega_0}{2}[e^{j\Omega_0 n + \theta + (\pi/2)} + e^{-j\Omega_0 n + \theta + (\pi/2)}]$$

$$= j \frac{\Omega_0}{2}[e^{j(\Omega_0 n + \theta)} - e^{-j(\Omega_0 n + \theta)}],$$

$$y[n] = -\Omega_0 \sin (\Omega_0 n + \theta)$$

It is apparent from this expression that $H(\Omega)$ is a discrete-time differentiator. A similar result holds for $-\pi \leq \Omega_0 \leq 0$. 

If $\Omega_0$ is outside the range $-\pi \leq \Omega_0 \leq \pi$, we can express $x[n]$ identically using a $\Omega_0$ within this range. For example,

$$x[n] = \cos \left( \frac{3\pi}{2} n + \theta \right)$$

$$= \cos \left( -\frac{\pi}{2} n + \theta \right),$$

$$y[n] = \frac{\pi}{2} \sin \left( -\frac{\pi}{2} n + \theta \right)$$

**S12.5**

(a) We see by examining $y_1[n]$ and $y_2[n]$ that $y_1[n]$ averages $x[n]$ and thus tends to suppress changes while $y_2[n]$ tends to suppress components that have not varied from $x[n-1]$ to $x[n]$. Therefore, the $y_1[n]$ system is lowpass and $y_2[n]$ is highpass.

(b) Taking the Fourier transforms yields

$$Y_1(\Omega) = X(\Omega) \left( \frac{1 + e^{-j\Omega}}{2} \right),$$

$$H_1(\Omega) = \frac{1}{2} \left( 1 + e^{-j\Omega} \right)$$

![Figure S12.5-1](image1)

$$Y_2(\Omega) = X(\Omega) \left( \frac{1 - e^{-j\Omega}}{2} \right),$$

$$H_2(\Omega) = \frac{1}{2} \left( 1 - e^{-j\Omega} \right)$$

![Figure S12.5-2](image2)
(a) By inspection we see that the impulse response is given by
\[ h_i[n] = \frac{1}{2N + 1} \sum_{k=-N}^{N} \delta[n - k] \]

(b) \( H_2(\Omega) = 1 - \frac{1}{2N + 1} \left[ \frac{\sin \left( \frac{2N + 1}{2} \right)}{\sin(\Omega/2)} \right] \)

(c) \[ |H_1(\Omega)| \]

(d) Zero and one crossings are at
\( \left( \frac{2\pi}{2N + 1} \right) k. \)

(d) \( H_3(\Omega) \) is an approximation to a highpass filter.

(a) From the specification that \( H(0) = 1 \), we know that
\[ H(\omega) = \frac{\alpha}{\alpha + j\omega} \]

(b) \[ |H(\omega)| = \frac{\alpha}{(\alpha^2 + \omega^2)^{1/2}}, \]
\[ \frac{\alpha}{(\alpha^2 + 4)^{1/2}} = \left| H(\omega) \right| \bigg|_{\omega = 2} \]
The low end specification is satisfied for $\alpha \geq 4$, as shown in Figure S12.7-1.

![Figure S12.7-1](image)

The high end specification is met for $\alpha \leq 6$, as shown in Figure S12.7-2.

![Figure S12.7-2](image)

The range of $\alpha$ such that the total specification is met is $4 \leq \alpha \leq 6$.

**Solutions to Optional Problems**

**S12.8**

The easiest method for solving this problem is to recognize that passing $x(t)$ through $H(\omega)$ is equivalent to performing

$$-2 \frac{dx(t)}{dt}$$

This is easily seen since

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} \, d\omega,$$

$$-2 \frac{dx(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} -2j\omega X(\omega)e^{j\omega t} \, d\omega$$

so

$$-2 \frac{dx(t)}{dt} \leftrightarrow -2j\omega X(\omega)$$
(a) \(-2 \frac{dx(t)}{dt} = -2 \frac{de^{jt}}{dt} = -2je^{jt} = y(t)\)

(b) \(-2 \frac{dx(t)}{dt} = -2 \frac{d((\sin \omega t)u(t))}{dt} = -2\omega_0(\cos \omega t)u(t)\)

(c) 
\[
X(\omega) = \frac{1}{j\omega(6 + j\omega)} = \frac{1}{j\omega} + \frac{1}{6 + j\omega},
\]
x(t) = \left[ \frac{1}{6} u(t) - \frac{1}{2} \right] - \frac{1}{6} e^{-6t}u(t)

\[-2 \frac{dx(t)}{dt} = -2 \left[ \frac{1}{6} \delta(t) + e^{-6t}u(t) - \frac{1}{6} e^{-6t}\delta(t) \right] = -2e^{-6t}u(t)\]

Alternatively, for this part it is perhaps simpler to use the fact that
\[
Y(\omega) = H(\omega)X(\omega) = \frac{-2j\omega}{j\omega(6 + j\omega)} = \frac{2}{6 + j\omega}
\]
so that \(y(t) = -2e^{-6t}u(t)\)

(d) 
\[
X(\omega) = \frac{1}{2 + j\omega}
\]
x(t) = \(e^{-2t}u(t)\)

\[-2 \frac{dx(t)}{dt} = -2[-2e^{-2t}u(t) + e^{-2t}\delta(t)] = 4e^{-2t}u(t) - 2\delta(t)\]

S12.9

(a) \(H(\Omega) = H_1(\Omega)e^{-j\Omega}\)

(i) \(H_1(\Omega)\) is real and even:

\(h[n] \leftrightarrow H_1(\Omega)\)

From Table 5.1 of the text (page 335), we see that the even part of \(h[n]\) has a Fourier transform that is the real part of \(H_1(\Omega)\). This result is easily verified:

\[
\sum_{n=-\infty}^{\infty} h[-n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} h[n]e^{j\Omega n} = \left( \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n} \right)^* = H^*_1(\Omega),
\]

so

\(\frac{1}{2}(h[n] + h[-n]) \leftrightarrow \frac{1}{2}[H_1(\Omega) + H^*_1(\Omega)],\)

\(Ev[h[n]] \leftrightarrow Re[H_1(\Omega)]\)

Now since

\(Re[H_1(\Omega)] = H_1(\Omega),\)

we have that \(Ev[h[n]] = h[n]\), i.e., \(h[n]\) is even, and therefore

\(h[n] = h[-n]\)
(ii) From Table 5.1, 
\[ x[n - n_0] \leftrightarrow e^{-j\Omega_0}, \]
so 
\[ H_1(\Omega)e^{-j\Omega M} \leftrightarrow h_1[n - M], \]
\[ h[n] = h_1[n - M] \]

(b) \( h_4[n] = h_4[-n] \)
Since \( h[n] = h_4[n - M] \),
\[ h[n + M] = h_4[n], \]
\[ h[M - n] = h_4[(M - n) - M] = h_4[-n], \]
but
\[ h_4[n] = h_4[-n] \rightarrow h[M - n] = h[M + n] \]

(c) \( h[M + n] = h[M - n] \) from part (b). Since \( h[n] \) is causal, \( h[M - n] = 0 \) for \( n > M \). But if \( h[M + n] = h[M - n] \), then 
\[ h[M + n] = 0 \quad \text{for } n > M, \]
so
\[ h[n] = 0 \quad \text{for } n > 2M \]

Summarizing, we have
\[ h[n] = 0 \quad \text{for } n < 0, n > 2M \]

S12.10

(a) 

Figure S12.10-1
(b) If the cutoff frequency $\Omega_c = \pi/N$, the total system is an identity system.

(c) $h[n] = \sum_{k=0}^{N-1} h[k] = \sum_{k=0}^{N-1} e^{j(2\pi k/N)}h[k]$

$$= \left[ \frac{1 - e^{j2\pi}}{1 - e^{j(2\pi k/N)}} \right] h[k],$$

$$h[n] = \begin{cases} Nh[n], & n = \text{an integer multiple of } N, \\ 0, & n \neq \text{an integer multiple of } N, \end{cases}$$

so $r[n]$ is as shown in Figure S12.10-2.

(d) $h_0[n] = \frac{1}{N}$, \quad $n = 0$,

$h_0[n] = 0$, \quad $n = \text{an integer multiple of } N$,

are the necessary and sufficient conditions.

\textbf{S12.11}

From the system diagram,

$Y(\omega) = X(\omega)[a - G(\omega)]$,

$H(\omega) = a - G(\omega)$

(a) $\forall \omega \in H(\omega)$ is 0 for all $\omega$.
(b) $|H(\omega)|$

$1 - \alpha$

$\alpha$

$1 - \alpha$

$0 \quad \omega_1 \quad \omega_2$

$\omega$

$\chi H(\omega)$

$\pi$

$0 \quad \omega_1 \quad \omega_2$

$\omega$

Figure S12.11-2

(c) $\chi H(\omega)$ is $\pi$ for all $\omega$.

| $H(\omega)$ |

$1 + \alpha$

$\alpha$

$\omega_1 \quad \omega_2$

$\omega$

Figure S12.11-3
Resource: Signals and Systems
Professor Alan V. Oppenheim

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