11 Discrete-Time Fourier Transform

Solutions to Recommended Problems

S11.1

(a) \[ X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \]
\[ = \sum_{n=-\infty}^{\infty} (\frac{1}{4})^n u[n]e^{-j\Omega n} \]
\[ = \sum_{n=0}^{\infty} (\frac{1}{4}e^{-j\Omega})^n \]
\[ = \frac{1}{1 - \frac{1}{4}e^{-j\Omega}} \]

Here we have used the fact that
\[ \sum_{n=0}^{\infty} a^n = \frac{1}{1 - a} \quad \text{for } |a| < 1 \]

(b) \[ x[n] = (a^n \sin \Omega_0 n) u[n] \]

We can use the modulation property to evaluate this signal. Since
\[ \sin \Omega_0 n \leftrightarrow \frac{2\pi}{2j} [\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)], \]
periodically repeated, then
\[ X(\Omega) = \frac{\pi}{2j} \left[ \frac{1}{1 - ae^{-j(\Omega - \Omega_0)}} - \frac{1}{1 - ae^{-j(\Omega + \Omega_0)}} \right] \]
periodically repeated.

(c) \[ X(\Omega) = \sum_{n=0}^{\infty} e^{-j\Omega n} \]
\[ = \frac{1 - e^{-j4\Omega}}{1 - e^{-j\Omega}}, \]
using the identity
\[ \sum_{n=0}^{N-1} a^n = \frac{1 - a^N}{1 - a} \]

Alternatively, we can use the fact that \( x[n] = u[n] - u[n - 4], \) so
\[ X(\Omega) = \frac{1}{1 - e^{-j\Omega}} - \frac{e^{-j4\Omega}}{1 - e^{-j\Omega}} = \frac{1 - e^{-j4\Omega}}{1 - e^{-j\Omega}} \]

(d) \[ x[n] = (\frac{1}{4})^n u[n + 2] \]
\[ = (\frac{1}{4})^n + 2(\frac{1}{4})^{-2} u[n + 2] \]
\[ = 16(\frac{1}{4})^{n+2} u[n + 2] \]

We know that
\[ 16 \left( \frac{1}{4} \right)^n u[n] \leftrightarrow \frac{16}{1 - \frac{1}{4}e^{-j\Omega}}, \]
so
\[ 16 \left( \frac{1}{4} \right)^{n+2} u[n + 2] \leftrightarrow \frac{16e^{j2\Omega}}{1 - \frac{1}{4}e^{-j\Omega}} \]
S11.2

(a) The difference equation \(y[n] - \frac{1}{2}y[n - 1] = x[n]\), which is initially at rest, has a system transfer function that can be obtained by taking the Fourier transform of both sides of the equation. This yields
\[Y(\Omega)(1 - \frac{1}{2}e^{-j\Omega}) = X(\Omega),\]
so
\[H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}}\]

(b) (i) If \(x[n] = \delta[n]\), then \(X(\Omega) = 1\) and
\[Y(\Omega) = H(\Omega)X(\Omega) = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}},\]
so
\[y[n] = (\frac{1}{2})^n u[n]\]

(ii) \(X(\Omega) = e^{-j\Omega n_0}\), so
\[Y(\Omega) = \frac{e^{-j\Omega n_0}}{1 - \frac{1}{2}e^{-j\Omega}}\]
and, using the delay property of the Fourier transform,
\[y[n] = (\frac{1}{2})^n u[n - n_0]\]

(iii) If \(x[n] = (\frac{1}{2})^n u[n]\), then
\[X(\Omega) = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}},\]
\[Y(\Omega) = \left(\frac{1}{1 - \frac{1}{2}e^{-j\Omega}}\right) \left(\frac{1}{1 - \frac{1}{2}e^{-j\Omega}}\right) = \frac{-2}{1 - \frac{1}{2}e^{-j\Omega}} + \frac{3}{1 - \frac{1}{2}e^{-j\Omega}},\]
so
\[y[n] = -2(\frac{1}{2})^n u[n] + 3(\frac{1}{2})^n u[n]\]

S11.3

(a) We are given a system with impulse response
\[h[n] = \left[\left(\frac{1}{2}\right)^n \cos \frac{\pi n}{2}\right] u[n]\]
The signal \(h[n] = (\frac{1}{2})^n u[n]\) has the Fourier transform
\[H_i(\Omega) = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}}\]
Using the modulation theorem, we have
\[H(\Omega) = \frac{1}{2} \left[\frac{1}{1 - \frac{1}{2}e^{-j(\Omega - \pi/2)}} + \frac{1}{1 - \frac{1}{2}e^{-j(\Omega + \pi/2)}}\right]\]

(b) We expect the system output to be a sinusoid modified in amplitude and phase. Using the results in part (a) and the fact that
\[x[n] = \frac{1}{2}e^{j(n/2)} + \frac{1}{2}e^{-j(n/2)},\]
we have

\[
H(\Omega) \bigg|_{\Omega = \pi/2} = \frac{1}{2} \left( \frac{1}{1 - e^{-j\pi/2}} + \frac{1}{1 + e^{-j\pi/2}} \right)
\]

\[
= \frac{1}{2} \left( 2 + \frac{2}{3} \right) = \frac{4}{3}
\]

\[
H(\Omega) \bigg|_{\Omega = -\pi/2} = H^*(\Omega) \bigg|_{\Omega = \pi/2} = \frac{4}{3}
\]

so

\[
y[n] = \frac{2}{3} e^{j(n\pi/2)} + \frac{2}{3} e^{-j(n\pi/2)}
\]

\[
= \frac{4}{3} \cos \frac{\pi}{2} n
\]

**S11.4**

(a) The use of the Fourier transform simplifies the analysis of the difference equation.

\[
y[n] + \frac{1}{4} y[n - 1] - \frac{1}{8} y[n - 2] = x[n] - x[n - 1],
\]

\[
Y(\Omega)(1 + \frac{1}{4} e^{-j\Omega} - \frac{1}{8} e^{-j2\Omega}) = X(\Omega)(1 - e^{-j\Omega}),
\]

\[
\frac{Y(\Omega)}{X(\Omega)} = H(\Omega) = \frac{1 - e^{-j\Omega}}{(1 + \frac{1}{4} e^{-j\Omega})(1 - \frac{1}{4} e^{-j\Omega})}
\]

We want to put this in a form that is easily invertible to get the impulse response \( h[n] \). Using a partial fraction expansion, we see that

\[
H(\Omega) = \frac{2}{1 + \frac{1}{4} e^{-j\Omega}} + \frac{-1}{1 - \frac{1}{4} e^{-j\Omega}}
\]

so

\[
h[n] = 2(-\frac{1}{4})^n u[n] - (\frac{1}{4})^n u[n]
\]

(b) At \( \Omega = 0 \), \( H(\Omega) = 0 \). At \( \Omega = \pi/4 \), \( H(\Omega) = 0.65 e^{j(1.22)} \). Since \( h[n] \) is real, \( H(\Omega) = H^*(-\Omega) \), so \( H(-\Omega) = H^*(\Omega) \) and \( H(-\pi/4) = 0.65 e^{-j(1.22)} \). Since \( H(\Omega) \) is periodic in \( 2\pi \),

\[
H\left(\frac{9\pi}{4}\right) = H\left(\frac{\pi}{4}\right) = 0.65 e^{j(1.22)}
\]

**S11.5**

(a) \( x[n] \) is an aperiodic signal with extent \([0, N - 1]\). The periodic signal

\[
\hat{y}[n] = \sum_{r=-\infty}^{\infty} x[n + rN]
\]

is periodic with period \( N \). To get the Fourier series coefficients for \( \hat{y}[n] \), we sum over one period of \( \hat{y}[n] \) to get

\[
a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk(2\pi/N)n}
\]
(b) The Fourier transform of \( x[n] \) is

\[
X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}
\]

\[
= \sum_{n=0}^{N-1} x[n] e^{-j\Omega n}
\]

since \( x[n] = 0 \) for \( n < 0, n > N - 1 \).

We can now easily see the relation between \( a_k \) and \( X(\Omega) \) since

\[
\left. \frac{1}{N} X(\Omega) \right|_{\Omega = \frac{2\pi k}{N}} = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\Omega(2\pi/N)n} = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n}
\]

Therefore,

\[
\frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n} = a_k
\]

**S11.6**

(a) | Signal Description | Transform |
--- | --- | --- |
Continuous time | Infinite duration | Periodic | I, III |
Continuous time | Infinite duration | Aperiodic | III |
Continuous time | Finite duration | Aperiodic | III, I* |
Discrete time | Infinite duration | Periodic | II, IV |
Discrete time | Infinite duration | Aperiodic | IV |
Discrete time | Finite duration | Aperiodic | IV, II* |

*Because these two signals are aperiodic, we know that they do not possess a Fourier series. However, since they are both finite duration, the Fourier series can be used to express a periodic signal that is formed by periodically replicating the finite-duration signal.*

(b) The discrete-time Fourier series has time- and frequency-domain duality. Both the analysis and synthesis equations are summations. The continuous-time Fourier transform has time- and frequency-domain duality. Both the analysis and synthesis equations are integrals.

(c) The discrete-time Fourier series and Fourier transform are periodic with periods \( N \) and \( 2\pi \) respectively.

**Solutions to Optional Problems**

**S11.7**

Because of the discrete nature of a discrete-time signal, the time/frequency scaling property does not hold. A result that closely parallels this property but does hold
for discrete-time signals can be developed. Define
\[ x_{nk}[n] = \begin{cases} \frac{x[n]}{k}, & \text{if } n \text{ is a multiple of } k, \\ 0, & \text{otherwise} \end{cases} \]

\( x_{nk}[n] \) is a "slowed-down" version of \( x[n] \) with zeros interspersed. By analysis in the frequency domain,

\[ X_{nk}(\Omega) = X(k\Omega), \]

which indicates that \( X_{nk}(\Omega) \) is compressed in the frequency domain.

S11.8

(a) \( X(\Omega - \Omega_0) \) is a shift in frequency of the spectrum \( X(\Omega) \). We will see later that this is the result of modulating \( x[n] \) with an exponential carrier. To derive the modification \( x_m[n] \), we use the synthesis equation:

\[ x_m[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega - \Omega_0)e^{jn\Omega} d\Omega \]

Changing variables so that \( \Omega - \Omega_0 = \Omega' \), we have

\[ x_m[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega')e^{j(n + \Omega_0)\Omega'} d\Omega' = x[n]e^{j\Omega_0n} \]

(b) Using the synthesis equation, we have

\[ \frac{1}{2\pi} \int_{2\pi} Re\{X(\Omega)e^{jn\Omega}\} d\Omega = \frac{1}{2\pi} \int_{2\pi} \frac{1}{2} [X(\Omega) + X^*(\Omega)]e^{jn\Omega} d\Omega \]

\[ = \frac{1}{2} x[n] + \frac{1}{2\pi} \left( \int_{2\pi} \frac{1}{2} X(\Omega)e^{-jn\Omega} d\Omega \right)^* \]

\[ = \frac{1}{2} [x[n] + x^*[-n]] \]

(c) \[ \frac{1}{2\pi} \int_{2\pi} Im\{X(\Omega)e^{jn\Omega}\} d\Omega = \frac{1}{2\pi} \int_{2\pi} \left[ \frac{X(\Omega) - X^*(\Omega)}{2j} \right] e^{jn\Omega} d\Omega \]

\[ = \frac{1}{2j} x[n] - \frac{1}{2\pi} \left( \frac{1}{2\pi} \int_{2\pi} X(\Omega)e^{-jn\Omega} d\Omega \right)^* \]

\[ = \frac{1}{2j} [x[n] - x^*[-n]] \]

(d) Since \( |X(\Omega)|^2 = X(\Omega)X^*(\Omega) \), we see that the inverse transform will be in the form of a convolution. Since

\[ \frac{1}{2\pi} \int_{2\pi} X^*(\Omega)e^{jn\Omega} d\Omega = \left( \frac{1}{2\pi} \int_{2\pi} X(\Omega)e^{-jn\Omega} d\Omega \right)^* \]

\[ = x^*[-n], \]

then

\[ \frac{1}{2\pi} \int_{2\pi} |X(\Omega)|^2e^{jn\Omega} d\Omega = x[n] \cdot x^*[-n] \]
We are given an LTI system with impulse response

\[ h[n] = \frac{\sin(\pi n/3)}{\pi n} \]

(a) We know from duality that \( H(\Omega) \) is a pulse sequence that is periodic with period \( 2\pi \). Suppose we assume this and adjust the parameters of the pulse so that

\[ \frac{1}{2\pi} \int H(\Omega)e^{jn\Omega} d\Omega = h[n] \]

Let \( a \) be the pulse amplitude and let \( 2W \) be the pulse width. Then

\[ \frac{a}{2\pi} \int_{-W}^{W} e^{jn\Omega} d\Omega = \frac{a}{2\pi} \left( \frac{e^{jwn} - e^{-jwn}}{jn} \right) \]

\[ = \frac{a}{2\pi} 2 \sin Wn \]

so \( a = 1 \) and \( W = \pi/3 \), as indicated in Figure S11.9-1.

(b) We know that

\[ \cos \frac{3\pi}{4} n \longleftrightarrow \pi \left[ \delta \left( \Omega - \frac{3\pi}{4} \right) + \delta \left( \Omega + \frac{3\pi}{4} \right) \right] \]

periodically repeated, and that multiplication by \((-1)^n\) shifts the periodic spectrum by \( \pi \), so the spectrum \( Y(\Omega) \) is as shown in Figure S11.9-2.

From Figures S11.9-1 and S11.9-2, we can see that

\[ Y(\Omega) = H(\Omega)X(\Omega) = X(\Omega) \]
Therefore,

\[ y[n] = x[n] = (-1)^n \cos \frac{3\pi n}{4} = \cos \frac{\pi n}{4} \]

S11.10

Here

\[ Y(\Omega) = 2X(\Omega) + e^{-j\Omega}X(\Omega) - \frac{dX(\Omega)}{d\Omega} \]

(a) (i) The system is linear because if

\[ x[n] = ax_1[n] + bx_2[n], \]

then

\[ y[n] = ay_1[n] + by_2[n], \]

where \( y_1[n] \) is obtained from \( x_1[n] \) via the given transfer function. The similar result applies for \( y_2[n] \).

(ii) The system is time-varying by the following argument.

If \( x[n] \rightarrow y[n] \), does \( x[n-1] \rightarrow y[n-1] \)?

The corresponding \( Y(\Omega) \) is

\[ 2e^{j\Omega}X(\Omega) + e^{-j\Omega}X(\Omega)e^{-j\Omega} + je^{-j\Omega}X(\Omega) \]

\[ \neq e^{-j\Omega} \left[ 2X(\Omega) + e^{-j\Omega}X(\Omega) - \frac{dX(\Omega)}{d\Omega} \right] \]

(iii) If \( x[n] = \delta[n] \), \( X(\Omega) = 1 \). Then

\[ Y(\Omega) = 2 + e^{-j\Omega}, \]

\[ y[n] = 2\delta[n] + \delta[n - 1] \]

S11.11

\[ \hat{x}[n] = \sum_{k=\langle N \rangle} a_k e^{j(k2\pi/N)n} \]

(a) If we multiply both sides of this equation by \( e^{-j(l2\pi/N)n} \) and sum over \( \langle N \rangle \), we obtain

\[ \sum_{n=\langle N \rangle} \hat{x}[n]e^{-j(l2\pi/N)n} = \sum_{k=\langle N \rangle} \sum_{n=\langle N \rangle} a_k e^{j(k-l)(2\pi/N)n} \]

If \( k \) is held fixed, the summation over \( \langle N \rangle \) is zero unless \( k = l \), which yields \( Na_l \). Thus

\[ a_l = \frac{1}{N} \sum_{n=\langle N \rangle} \hat{x}[n]e^{-j(l2\pi/N)n} \]
and therefore

\[ a_k = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-j \frac{2\pi k}{N} n} \]

\[ x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j \Omega n} d\Omega \]

\( \text{(i)} \) By multiplying both sides by \( e^{-j \Omega n} \) and summing over all \( n \), we have

\[ \sum_{n=-\infty}^{\infty} x[n] e^{-j \Omega n} = \frac{1}{2\pi} \int_{2\pi} X(\Omega) \sum_{n=-\infty}^{\infty} e^{j \Omega n} d\Omega \]

\[ \sum_{n=-\infty}^{\infty} e^{j \Omega n} = \sum_{n=-\infty}^{\infty} a_n e^{j \frac{2\pi (\Omega - \Omega_1)}{T} n} , \]

where \( T = 2\pi \) and \( a_n = 1 \). The periodic function represented by this series is a periodic impulse train with period \( T = 2\pi \), so

\[ \sum_{n=-\infty}^{\infty} e^{j \Omega n} = 2\pi \sum_{n=-\infty}^{\infty} \delta(\Omega - \Omega_1 + 2\pi n) \]

\( \text{(ii)} \) Only a single impulse in the train appears in the integration interval of one period. So

\[ \frac{1}{2\pi} \int_{2\pi} X(\Omega) \sum_{n=-\infty}^{\infty} e^{j \Omega n} d\Omega = X(\Omega_1 + 2\pi n) \]

\[ = X(\Omega_1) \]

Therefore, the analysis formula for aperiodic discrete signals has been verified to be analogous to the analysis formula in part (a).

\[ X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j \Omega n} \]

\( \text{S11.12} \)

(a) The Fourier transform of \( e^{j k (2\pi / N)n} \) can be performed by inspection using the synthesis formula

\[ e^{j \frac{2\pi k}{N} n} = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j \Omega n} d\Omega , \]

\[ X(\Omega) = 2\pi \delta \left( \Omega - \frac{2\pi k}{N} \right) , \quad |\Omega| < \pi \]

and since we know that \( X(\Omega) \) is periodic in \( \Omega = 2\pi \), we have

\[ e^{j k (2\pi / N)n} \quad \mathcal{F} \quad 2\pi \sum_{m=-\infty}^{\infty} \delta \left( \Omega - \frac{2\pi k}{N} + 2\pi m \right) \]

(b) By using superposition and the result in part (a), we have

\[ \sum_{k=(N)} a_k e^{j k (2\pi / N)n} \quad \mathcal{F} \quad \sum_{m=-\infty}^{\infty} 2\pi \sum_{k=(N)} a_k \delta \left( \Omega - \frac{2\pi k}{N} + 2\pi m \right) \]
(c) We can change the double summation to a single summation since \( a_k \) is periodic:

\[
\sum_{n=\infty}^{\infty} 2\pi \sum_{k=\{N\}} a_k \delta \left( \Omega - \frac{2\pi k}{N} + 2\pi n \right) = 2\pi \sum_{k=\{N\}} a_k \delta \left( \Omega - \frac{2\pi k}{N} \right)
\]

So we have established the Fourier transform of a periodic signal via the use of a Fourier series:

\[
\tilde{x}[n] = \sum_{k=\{N\}} a_k e^{j2\pi k n/N} \leftrightarrow 2\pi \sum_{k=\{N\}} a_k \delta \left( \Omega - \frac{2\pi k}{N} \right)
\]

(d) We have

\[
\tilde{x}[n] = \sum_{k=\{N\}} x[n - kN] \leftrightarrow \sum_{k=-\infty}^{\infty} X(\Omega) e^{-j2\pi k N}
\]

As in S11.11(b)(ii), we can show that

\[
\sum_{k=-\infty}^{\infty} e^{-j2\pi k N} = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta \left( \Omega - \frac{2\pi k}{N} \right)
\]

Therefore,

\[
\tilde{x}[n] \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} \frac{1}{N} X(\Omega) \delta \left( \Omega - \frac{2\pi k}{N} \right) = 2\pi \sum_{k=-\infty}^{\infty} \frac{1}{N} X \left( \frac{2\pi k}{N} \right) \delta \left( \Omega - \frac{2\pi k}{N} \right)
\]

Comparing with the result of part (c), we see that

\[
a_k = \frac{1}{N} X(\Omega) \bigg|_{\Omega = (2\pi k)/N}
\]
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