10 Discrete-Time Fourier Series

Solutions to Recommended Problems

S10.1

The output of a discrete-time linear, time-invariant system is given by

\[ y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n - k], \]

where \( h[n] \) is the impulse response and \( x[n] \) is the input. By substitution, we have the following.

(a) \[ y[n] = \sum_{k=0}^{\infty} \left( \frac{1}{2} \right)^k e^{jn(n-k)} = e^{jn} \sum_{k=0}^{\infty} \left( \frac{e^{-jn}}{2} \right)^k \]

\[ = \frac{e^{jn}}{1 - \frac{1}{2}e^{-jn}} = \frac{2}{3} (-1)^n \]

(b) \[ y[n] = \sum_{k=0}^{\infty} \left( \frac{1}{2} \right)^k e^{j\pi(n-k)/4} = e^{j\pi n/4} \sum_{k=0}^{\infty} \left( \frac{e^{-j\pi/4}}{2} \right)^k \]

\[ = \frac{e^{j\pi n/4}}{1 - \frac{1}{2}e^{-j\pi/4}} \]

(c) \[ y[n] = \sum_{k=0}^{\infty} \left( \frac{1}{2} \right)^k \left[ \frac{1}{2} e^{j\pi/8} \delta[n-(n-k)/4] + \frac{1}{2} e^{-j\pi/8} \delta[-j\pi/4] \right], \]

where we have used Euler's relation

\[ = \frac{1}{2} e^{j\pi/8} e^{j\pi n/4} \sum_{k=0}^{\infty} \left[ \frac{e^{-j\pi/4}}{2} \right]^k + \frac{1}{2} e^{-j\pi/8} e^{-j\pi n/4} \sum_{k=0}^{\infty} \left[ \frac{e^{j\pi/4}}{2} \right]^k \]

\[ = \frac{1}{2} e^{jn(n/4) + j(n/8)} + \frac{1}{2} e^{-j(n/4) + j(n/8)} \]

\[ = \cos \left( \frac{\pi n + \pi}{8} \right) - \frac{1}{2} \cos \left( \frac{\pi n + 3\pi}{8} \right) \]

\[ = \frac{5}{4} - \cos \left( \frac{\pi}{4} \right) \]

S10.2

(a) \( \hat{x}[n] = 1 + \sin \left( \frac{2\pi n}{10} \right) \)

To find the period of \( \hat{x}[n] \), we set \( \hat{x}[n] = \hat{x}[n + N] \) and determine \( N \). Thus

\[ 1 + \sin \left( \frac{2\pi n}{10} \right) = 1 + \sin \left( \frac{2\pi}{10} (n + N) \right) \]

\[ = 1 + \sin \left( \frac{2\pi}{10} n + \frac{2\pi}{10} N \right) \]

Since

\[ \sin \left( \frac{2\pi}{10} n + 2\pi \right) = \sin \left( \frac{2\pi}{10} n \right), \]
the period of \( \tilde{x}_1[n] \) is 10. Similarly, setting \( \tilde{x}_2[n] = \tilde{x}_2[n + N] \), we have

\[
1 + \sin \left( \frac{20\pi}{12} n + \frac{\pi}{2} \right) = 1 + \sin \left[ \frac{20\pi}{12} \left( n + N \right) + \frac{\pi}{2} \right] = 1 + \sin \left( \frac{20\pi}{12} n + \frac{\pi}{2} + \frac{20\pi}{12} N \right)
\]

Hence, for \( \frac{20\pi}{12} \pi N \) to be an integer multiple of \( 2\pi \), \( N \) must be 6.

(b) \( \tilde{x}_1[n] = 1 + \sin \left( \frac{2\pi n}{10} \right) \)

Using Euler’s relation, we have

\[
x_1[n] = 1 + \frac{1}{2j} e^{j(2\pi/10)n} - \frac{1}{2j} e^{-j(2\pi/10)n}
\]

Note that the Fourier synthesis equation is given by

\[
x_1'[n] = \sum_{k=(N)} a_k e^{j(2\pi/N)n}
\]

where \( N = 10 \). Hence, by inspection of eq. \( \text{(S10.2-1)} \), we see that

\[
a_0 = 1, \quad a_{-1} = -\frac{1}{2j},
\]

\[
a_{11} = \frac{1}{2j}, \quad \text{and}
\]

\[
a_{1k} = 0, \quad 2 \leq k \leq 8,
\]

\[
-8 \leq k \leq -2
\]

Similarly,

\[
\tilde{x}_2[n] = 1 + \frac{1}{2j} e^{j(\pi/2)e^{-j(2\pi/12)n}} - \frac{1}{2j} e^{-j(\pi/2)e^{j(2\pi/12)n}}
\]

Therefore, \( N = 12 \).

\[
a_{20} = 1, \quad a_{-1} = -\frac{e^{-j(\pi/2)}}{2j} = \frac{1}{2j}, \quad a_{21} = \frac{1}{2j} e^{j(\pi/2)} = \frac{1}{2}, \quad \text{and}
\]

\[
a_{2\pm 2}, \ldots, a_{2\pm 11} = 0
\]

(c) The sequence \( a_{1k} \) is periodic with period 10 and \( a_{2k} \) is periodic with period 12.

S10.3

The Fourier series coefficients can be expressed as the samples of the envelope

\[
a_k = \frac{1}{N} \cdot \frac{\sin((2N + 1)\Omega/2)}{\sin(\Omega/2)} \bigg|_{\Omega = 2\pi k/N} \quad \text{where } N_i = 1 \text{ (see Example 5.3 on page 302 of the text)}
\]

(a) For \( N = 6 \),

\[
a_k = \frac{1}{6} \cdot \frac{\sin \left( \frac{3}{2} \frac{2\pi k}{6} \right)}{\sin \left( \frac{1}{2} \frac{2\pi k}{6} \right)} = \frac{1}{6} \cdot \frac{\sin \left( \frac{\pi k}{2} \right)}{\sin \left( \frac{\pi k}{6} \right)}
\]
(b) For $N = 12$,
\[
    a_k = \frac{1}{12} \sin \left[ \frac{3}{2} \left( \frac{2\pi k}{12} \right) \right] = \frac{1}{12} \sin \left( \frac{\pi k}{4} \right)
\]

(c) For $N = 60$,
\[
    a_k = \frac{1}{60} \sin \left[ \frac{3}{2} \left( \frac{2\pi k}{60} \right) \right] = \frac{1}{60} \sin \left( \frac{\pi k}{20} \right)
\]

**S10.4**

(a) The discrete-time Fourier transform of the given sequence is
\[
    X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \\
    = \frac{1}{3}e^{j\Omega} + 1 + \frac{1}{3}e^{-j\Omega} \\
    = 1 + \cos \Omega
\]

$X(\Omega)$ is sketched in Figure S10.4.

(b) The first sequence can be thought of as
\[
    y_1[n] = x[n] \ast \left[ \sum_{k=-\infty}^{\infty} \delta[n - 3k] \right]
\]

Hence
\[
    Y_1(\Omega) = X(\Omega) \frac{2\pi}{3} \sum_{k=-\infty}^{\infty} \delta \left( \Omega - \frac{2\pi k}{3} \right)
\]

Therefore, the Fourier series of $y_1[n]$ is given by
\[
    a_k = \frac{1}{2\pi} Y_1 \left( \frac{2\pi}{3} k \right) = \frac{1}{3} \left( 1 + \cos \frac{2\pi k}{3} \right), \quad \text{for all } k
\]
The second sequence is given by
\[ y_2[n] = x[n] \cdot \left[ \sum_{k=-\infty}^{\infty} \delta[n - 5k] \right] \]

Similarly, the Fourier series of this sequence is given by
\[ a_k = \frac{1}{5} \left[ 1 + \cos \left( \frac{2\pi k}{5} \right) \right], \quad \text{for all } k \]

This result can also be obtained by using the fact that the Fourier series coefficients are proportional to equally spaced samples of the discrete-time Fourier transform of one period (see Section 5.4.1 of the text, page 314).

**S10.5**

(a) The given relation
\[ x[n] = \sum_{k=0}^{3} a_k e^{j(2\pi/4)n} \]
results in the following set of equations
\[ a_0 + a_1 + a_2 + a_3 = x[0] = 1, \]
\[ a_0 + a_1 e^{j(\pi/2)} + a_2 e^{j\pi} + a_3 e^{j(3\pi/2)} = x[1] = 0, \]
\[ a_0 + a_1 e^{j\pi} + a_2 e^{j3\pi} + a_3 e^{j5\pi} = x[2] = 2, \]
\[ a_0 + a_1 e^{j(3\pi/2)} + a_2 e^{j(3\pi)} + a_3 e^{j(9\pi/2)} = x[3] = -1 \]

The preceding set of linear equations can be reduced to the form
\[ a_0 + a_1 + a_2 + a_3 = 1, \]
\[ a_0 + ja_1 - a_2 - ja_3 = 0, \]
\[ a_0 - a_1 + a_2 - a_3 = 2, \]
\[ a_0 - ja_1 - a_2 + ja_3 = -1 \]

Solving the resulting equations, we get
\[ a_0 = \frac{1}{2}, \quad a_1 = -\frac{1+j}{4}, \quad a_2 = +1, \quad a_3 = -\frac{1-j}{4} \quad \text{(S10.5-1)} \]

By the discrete-time Fourier series analysis equation, we obtain
\[ a_k = \frac{1}{4} \left[ 1 + 2e^{-j\pi k} - e^{-j(3\pi/2)k} \right], \]

which is the same as eq. (S10.5-1) for \( 0 \leq k \leq 3 \).

**S10.6**

(a) \( a_k = a_{k+10} \) for all \( k \) is true since \( x[n] \) is periodic with period 10.

(b) \( a_k = a_{-k} \) for all \( k \) is false since \( x[n] \) is not even.

(c) \( a_k e^{j(2\pi/5)} \) is real. This statement is true because it would correspond to the Fourier series of \( x[n] + 2 \), which is a purely real and even sequence.

(d) \( a_0 = 0 \) is true since the sum of the values of \( x[n] \) over one period is zero.
Solutions to Optional Problems

S10.7

The Fourier series coefficients of $x[n]$, which is periodic with period $N$, are given by

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk(2\pi/N)n}$$

For $N = 8$,

$$a_k = \frac{1}{8} \sum_{n=0}^{7} x[n] e^{-jk(\pi/4)n} \quad (S10.7-1)$$

(a) We are given that

$$a_k = \cos \left( \frac{k\pi}{4} \right) + \sin \left( \frac{3k\pi}{4} \right), \quad (S10.7-2)$$

Hence, by comparing eqs. (S10.7-1) and (S10.7-2) we can immediately write

$$x[n] = 4\delta[n - 1] + 4\delta[n - 7] - 4j\delta[n - 3] + 4\delta[n - 5], \quad 0 \leq n \leq 7$$

(b) $x[n] = \sum_{k=0}^{7} a_k e^{j(2\pi/8)kn} = \sum_{k=0}^{7} a_k e^{j(\pi/4)n}$

$$= \sum_{k=0}^{6} \left[ \frac{1}{2j} e^{j(\pi/3)} - \frac{1}{2j} e^{-j(\pi/3)} \right] e^{j(\pi/4)n}$$

$$= \frac{1}{2j} \sum_{k=0}^{6} e^{j(k(\pi/3)+(n/4))} - \frac{1}{2j} \sum_{k=0}^{6} e^{-j(k(\pi/3)-(n/4))}$$

$$= \frac{1}{2j} \left[ 1 - e^{j((7n/4)+(7\pi/3))} \right] - \frac{1}{2j} \left[ 1 - e^{j((7n/4)-(7\pi/3))} \right]$$

(c) $x[n] = \sum_{k=0}^{7} a_k e^{j(2\pi/8)kn} = \sum_{k=0}^{7} a_k e^{j(\pi/4)n}$

$$= 1 + e^{j(\pi/4)n} + e^{j(3\pi/4)n} + e^{j(\pi/4)n} + e^{j(5\pi/4)n} + e^{j(\pi/4)n}$$

$$= 1 + (-1)^n + 2 \cos \left( \frac{\pi}{4} n \right) + 2 \cos \left( \frac{3\pi}{4} n \right), \quad 0 \leq n \leq 7$$

(d) Using an analysis similar to that in part (c), we find

$$x[n] = 2 + 2 \cos \left( \frac{\pi}{4} n \right) + \cos \left( \frac{\pi}{2} n \right) + \frac{1}{2} \cos \left( \frac{3\pi}{4} n \right), \quad 0 \leq n \leq 7$$

S10.8

The impulse response of the LTI system is

$$h[n] = (\frac{1}{2})^{|n|}$$
The discrete-time Fourier transform of $h[n]$ is

$$H(\Omega) = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n e^{-j\Omega n} + \sum_{n=-\infty}^{0} \left( \frac{1}{2} \right)^{-n} e^{-j\Omega n} - 1$$

$$= \frac{1}{1 - \frac{1}{2}e^{-j\Omega}} + \frac{1}{1 - \frac{1}{2}e^{j\Omega}} - 1$$

$$= \frac{3}{5 - 4 \cos \Omega}$$

(a) (i) $x[n] = \sin \left( \frac{3\pi}{4} n \right) = \frac{1}{2j} e^{j(3\pi/4)n} - \frac{1}{2j} e^{-j(3\pi/4)n}$

The period of $x[n]$ is

$$\sin \left( \frac{3\pi}{4} n \right) = \sin \left[ \frac{3\pi}{4} (n + N) \right]$$

Thus

$$\sin \left( \frac{3\pi}{4} n \right) = \sin \left( \frac{3\pi}{4} n + \frac{3\pi}{4} N \right)$$

We set $3\pi N/4 = 2\pi m$ to get $N = 8$ ($m = 3$). Hence, the period is 8.

$$x[n] = \sum_{k=0}^{7} a_k e^{j(k\pi/8)n}$$

Therefore,

$$a_3 = \frac{1}{2j^2} = a_5^*$$

All other coefficients $a_k$ are zero. By the convolution property, the Fourier series representation of $y[n]$ is given by $b_k$, where

$$b_k = a_k H(\Omega) \bigg|_{0 = (2\pi k)/8}$$

Thus

$$b_3 = \frac{1}{2j} \frac{3}{5 - 4 \cos(3\pi/4)} = b_5^*$$

All other $b_k$ are zero in the range $0 \leq k \leq 7$.

(ii) $\ddot{x}[n] = \sum_{k=-\infty}^{\infty} \delta[n - 4k]$

The Fourier series of $\ddot{x}[n]$ is

$$a_k = \frac{1}{4} \sum_{n=0}^{3} \ddot{x}[n] e^{-jk(2\pi/4)m} = \frac{1}{4}, \text{ for all } k$$

And the Fourier series of $\ddot{y}[n]$ is

$$b_k = a_k H(\Omega) \bigg|_{0 = \pi k/2}$$

$$= \frac{1}{4} \frac{3}{5 - 4 \cos((\pi/2)k)} = \frac{3}{20} \text{ for all } k$$
(iii) The Fourier series of $\tilde{x}[n]$ is
$$a_k = \frac{1}{6} \left[ 1 + 2 \cos \left( \frac{\pi}{3} k \right) \right], \quad 0 \leq k \leq 5$$
and the Fourier series of $\tilde{y}[n]$ is
$$b_k = a_k H(\Omega) \bigg|_{\Omega = (\pi/3)k} = \frac{1}{6} \left[ 1 + 2 \cos \left( \frac{\pi}{3} k \right) \right] \frac{3}{5 - 4 \cos((\pi/3)k)}$$

(iv) $x[n] = j^n + (-1)^n$

The period of $\tilde{x}[n]$ is 4. $x[n]$ can be rewritten as
$$x[n] = [e^{i(\pi/2)n} + e^{i\pi n}]$$
$$= \sum_{k=0}^3 a_k e^{j(2\pi/4)n}$$

Hence,
$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 1, \quad a_3 = 0$$

Therefore, $b_0 = b_3 = 0$ and
$$b_1 = \frac{3}{5 - 4 \cos(\pi/2)} = \frac{3}{5},$$
$$b_2 = \frac{3}{5 - 4 \cos \pi} = \frac{3}{9}$$

(b) $h[n]$ is sketched in Figure S10.8.

(b) $h[n]$ is sketched in Figure S10.8.

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j2\Omega n} = -e^{j2\Omega} - e^{j\Omega} + 1 + e^{-j\Omega} + e^{-j2\Omega},$$
$$H(\Omega) = 1 - 2j \sin \Omega - 2j \sin 2\Omega$$

It follows from part (a):

(i) $$b_3 = \frac{1}{2j} H(\Omega) \bigg|_{\Omega = 3\pi/4} = \frac{1}{2j} \left( - \sin \frac{3\pi}{4} - \sin \frac{3\pi}{2} \right) = b_3^*$$

All other coefficients $b_k$ are zero, in the range $0 \leq k \leq 7$. 
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S10.8

(ii) \[ b_k = \frac{1}{2} H(\Omega) \bigg|_{\Omega = \pi k/2} = \frac{1}{4} - \frac{j}{2} \sin \frac{\pi k}{2} \]
\[ = \frac{1}{4} - \frac{j}{2} \sin \pi k = \frac{1}{4} - \frac{j}{2} \sin \frac{\pi k}{2} \]

(iii) \[ b_k = \frac{1}{6} \left[ 1 + 2 \cos \left( \frac{\pi}{3} k \right) \right] H(\Omega) \bigg|_{\Omega = \pi k/3} \]

(iv) \[ b_0 = 0, \]
\[ b_1 = H(\Omega) \bigg|_{\Omega = \pi/2} = 1 - 2j, \]
\[ b_2 = H(\Omega) \bigg|_{\Omega = \pi} = 1, \]
\[ b_3 = 0 \]

S10.9

\( x[n] \xrightarrow{\mathcal{F}} a_k \)

(a) \( x[n - n_0] \xrightarrow{\mathcal{F}} a_k e^{-j(2\pi/N)n_0} \)
(b) \( x[n] - x[n - 1] \xrightarrow{\mathcal{F}} a_k \left[ 1 - e^{-j(2\pi/k)} \right] \)
(c) \( x[n] - x \left[ n - \frac{N}{2} \right] \xrightarrow{\mathcal{F}} a_k \left( 1 - e^{-jk} \right), \quad N \) even

\[ = \begin{cases} 
0, & k \text{ even}, \\
2a_k, & k \text{ odd} 
\end{cases} \]

(d) \( x[n] + x \left[ n + \frac{N}{2} \right], \quad \text{period} \frac{N}{2} \)
\[ a_k = \frac{2}{N} \sum_{n=0}^{(N/2)-1} \left[ x[n] + x \left[ n + \frac{N}{2} \right] \right] e^{-j(4\pi/N)n} \]
\[ = 2a_{2k} \]

(e) \[ a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[-n] e^{-j(2\pi/N)n}, \]
\[ a_k^* = \frac{1}{N} \sum_{n=0}^{N-1} x[-n] e^{-j(2\pi/N)n} \]
\[ = \frac{1}{N} \sum_{n=0}^{-N+1} x[n] e^{-j(2\pi/N)n} = a_k \]

Therefore, \( a_k = a_k^* \).

S10.10

(a) \[ \tilde{w}[n] = \tilde{x}[n] + \tilde{y}[n], \]
\[ \tilde{w}[n + NM] = \tilde{x}[n + NM] + \tilde{y}[n + NM] \]
\[ = \tilde{x}[n] + \tilde{y}[n] \]
\[ = \tilde{w}[n] \]

Hence, \( \tilde{w}[n] \) is periodic with period \( NM \).
(b) \[ c_k = \frac{1}{NM} \sum_{n=0}^{NM-1} \hat{x}[n] e^{-j(2\pi / NM)n} + \frac{1}{NM} \sum_{n=0}^{NM-1} \hat{y}[n] e^{-j(2\pi / NM)n} \]

\[ = \frac{1}{NM} \sum_{n=0}^{NM-1} \hat{x}[n] e^{-j(2\pi / NM)n} + \frac{1}{NM} \sum_{n=0}^{NM-1} \hat{y}[n] e^{-j(2\pi / NM)n} \]

\[ = \frac{1}{NM} \sum_{n=0}^{NM-1} \hat{x}[n] e^{-j(2\pi / NM)(n + N)} + \frac{1}{NM} \sum_{n=0}^{NM-1} \hat{y}[n] e^{j(2\pi / NM)(n + 1M)} \]

\[ \begin{cases} 
\frac{1}{N} a_{k/M} + \frac{1}{M} b_{k/N}, & \text{for } k \text{ a multiple of } M \text{ and } N, \\
\frac{1}{N} a_{k/M}, & \text{for } k \text{ a multiple of } M, \\
\frac{1}{M} b_{k/N}, & \text{for } k \text{ a multiple of } N, \\
0, & \text{otherwise}
\end{cases} \]

S10.11

(a) \[ \hat{x}[n] = \sin \left[ \frac{\pi(n - 1)}{4} \right] \]

To find the period, we set \( \hat{x}[n] = \hat{x}[n + N] \). Thus,

\[ \sin \left[ \frac{\pi(n - 1)}{4} \right] = \sin \left[ \frac{\pi(n + N - 1)}{4} \right] = \sin \left[ \frac{\pi(n - 1) + \pi N}{4} \right] \]

Let \( \pi N / 4 = 2\pi i \), when \( i \) is an integer. Then \( N = 8 \) and

\[ \hat{x}[n] = \frac{1}{2j} e^{j\pi(n - 1)/4} - \frac{1}{2j} e^{-j\pi(n - 1)/4} \]

\[ = \frac{1}{2j} e^{-j\pi/4} e^{j\pi n/4} - \frac{1}{2j} e^{j\pi/4} e^{-j\pi n/4} \]

Therefore,

\[ a_1 = \frac{e^{-j\pi/4}}{2j}, \quad a_7 = \frac{e^{j\pi/4}}{2j} \]

All other coefficients \( a_k \) are zero, in the range \( 0 \leq k \leq 7 \). The magnitude and phase of \( a_k \) are plotted in Figure S10.11-1.
(b) The period $N = 21$ and the Fourier series coefficients are

$$a_7 = a_{14} = \frac{1}{2}, \quad a_3 = a_{18}^* = \frac{1}{2j}$$

The rest of the coefficients $a_k$ are zero. The magnitude and phase of $a_k$ are given in Figure S10.11-2.

(c) The period $N = 8$.

$$a_3 = a_5^* = \frac{1}{2}e^{-j\pi/3}$$

The rest of the coefficients $a_k$ are zero. The magnitude and phase of $a_k$ are given in Figure S10.11-3.
Resource: Signals and Systems
Professor Alan V. Oppenheim

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